Background knowledge – Trigonometry with right angled triangles

Contents:

INTRODUCTION

Trigonometry is the study of the relationship between lengths and angles of geometrical figures.

We can apply trigonometry in engineering, astronomy, architecture, navigation, surveying, the building industry and in many branches of applied science.

HISTORICAL NOTE

Astronomy leads to the development of trigonometry

The Greek astronomer **Hipparchus** (140 BC) is credited with being the originator of trigonometry. To aid his calculations regarding astronomy, he produced a table of numbers in which the lengths of chords of a circle were related to the length of the radius.

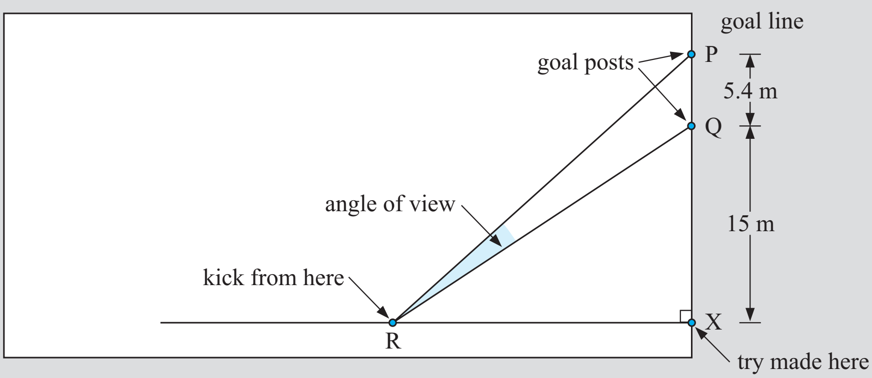
Ptolemy, another great Greek astronomer of the time, extended this table in his major published work *Almagest* which was used by astronomers for the next 1000 years. In fact, much of Hipparchus' work is known through the writings of Ptolemy. These writings found their way to Hindu and Arab scholars.

Aryabhata, a Hindu mathematician in the 6th Century AD, drew up a table of the lengths of half-chords of a circle with radius one unit. After completing this chapter you will see that the length of the half-chord is sine θ. So Aryabhata actually drew up the first table of sine values.

In the late 16th century, **Rhaeticus** produced comprehensive and remarkably accurate tables of all six trigonometric ratios (you will learn about three of these in this chapter). These involved a tremendous number of tedious calculations, all without the aid of calculators or computers!

OPENING PROBLEM

After a try is scored in a Rugby game Ray O'Farrell makes a place kick for goal to earn extra points. This kick must be taken from a point on an imaginary line which is perpendicular to the goal line out from the point where the try was made.



Later in this chapter you will be able to answer these questions:

1 If Ray is 30 m from the goal line, how far is he from the nearer goal post Q and what is the angle of view to the goal posts that Ray faces?

2 Can you use a scale diagram to check your answers to **1**, and to what degree of accuracy would your answer be?

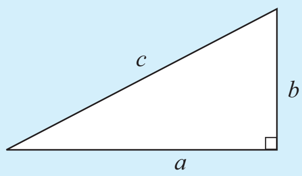
3 If Ray is x m from the goal line find an expression for the angle of view θ^o, using the tangent ratio.

4 Find how far Ray should place the ball from the goal line to maximise the angle of view which in theory would maximise his chance of kicking the goal.

A PYTHAGORAS' RULE (REVIEW)

The **Pythagoras' Rule** is:

In a right angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides. i.e., $c^2 = a^2 + b^2$.



Reminder: The **hypotenuse** is always the *longest side* and is *opposite the right angle.*

This theorem, known to the ancient Greeks, is particularly valuable in that:

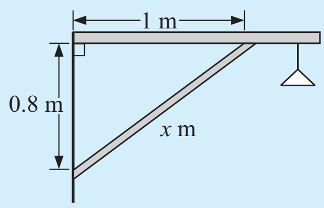
- if we know the lengths of any two sides of a right angled triangle then we can calculate the length of the third side
- if we know the lengths of the three sides then we can determine whether or not the triangle is right angled.

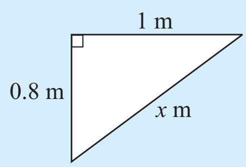
The second statement here relies on the **converse of Pythagoras' Rule**, which is:

If a triangle has sides of length a, b and c units and $a^2 + b^2 = c^2$ say, then the triangle is right angled and its hypotenuse is c units long.

Example 1

Find the unknown length in:





$x^2 = 0.8^2 + 1^2$

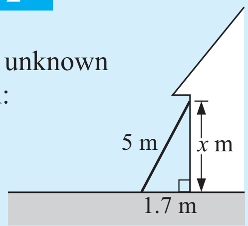
$\therefore \quad x = \sqrt{(0.8^2 + 1^2)}$

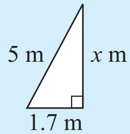
$\therefore \quad x = 1.2806......$

So, the length is 1.28 m.

Example 2

Find the unknown length in:





$x^2 + 1.7^2 = 5^2$

$\therefore \quad x^2 = 5^2 - 1.7^2$

$\therefore \quad x = \sqrt{(5^2 - 1.7^2)}$

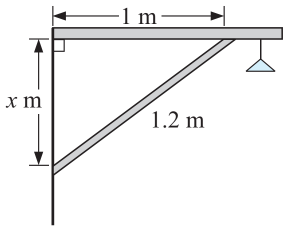
$\therefore \quad x = 4.7021....$

So, the length is 4.70 m.

EXERCISE A

1 Find, correct to 3 significant figures, the value of x in:

a



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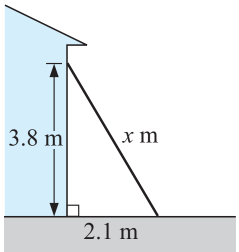
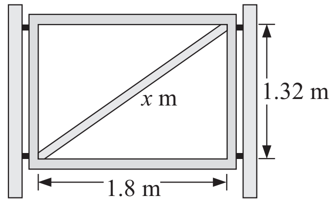
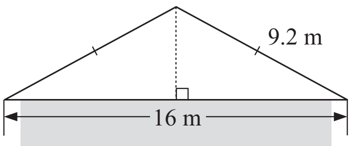


c



2 How high is the roof above the walls in the following roof structures?

a



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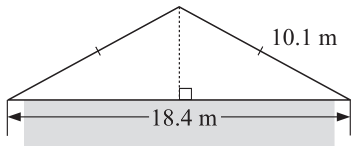
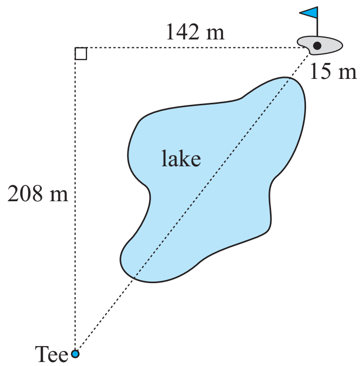


3 Bob is about to tee off on the sixth, a par 4 at the Royal Golf Club. If he chooses to hit over the lake, directly at the flag, how far must he hit the ball to clear the lake, given that the pin is 15 m from the water's edge?



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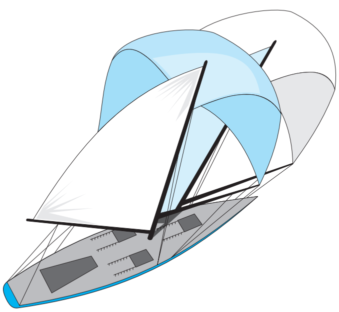


A sailing ship sails 46 km North then 74 km East.

a Draw a fully labelled diagram of the ship's course.

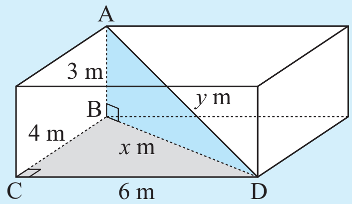
b How far is the ship from its starting point?

B PYTHAGORAS' RULE IN 3-D PROBLEMS

The rule of Pythagoras is often used twice in 3-D problem solving.

Example 3

A room is 6 m by 4 m at floor level and the floor to ceiling height is 3 m. Find the distance from a floor corner point to the opposite corner point on the ceiling.



The required distance is AD. We join BD.

In ΔBCD, $x^2 = 4^2 + 6^2$ {Pythagoras}

$\therefore\ x^2 = 16 + 36 = 52$

In ΔABD $y^2 = x^2 + 3^2$

$\therefore\ y^2 = 52 + 9 = 61$

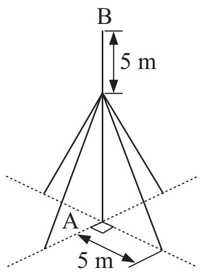
$\therefore\ y = \sqrt{61} \doteqdot 7.81$

i.e., the required distance is 7.81 m.

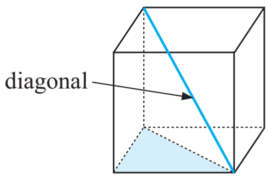
EXERCISE B

1 A pole AB is 16 m tall above the ground. At a point 5 m below B, four wires are connected from the pole to the ground.

Each wire is pegged to the ground 5 m from the base of the pole. What is the total length of the wire needed given that a total of 2 m extra is needed for tying?



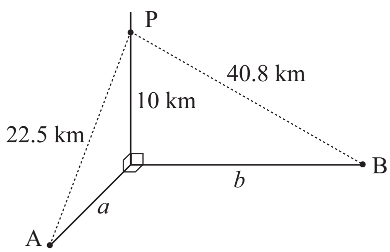
2 A cube has sides of length 10 cm. Find the length of a diagonal of the cube.



3 A room is 7 m by 4 m and has a height of 3 m. Find the distance from a corner point on the floor to the opposite corner of the ceiling.

4 A pyramid of height 40 m has a square base with edges 50 m. Determine the length of the slant edges.

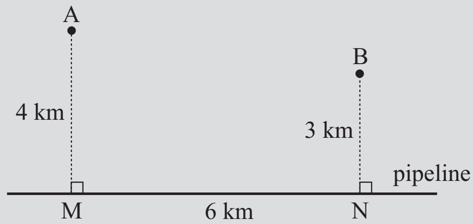
5



An aeroplane P is flying at an altitude of 10 000 m. The pilot sees two ships A and B. Ship A is due South of P and 22.5 km away (in a direct line). Likewise ship B is due East and 40.8 km from P. Find the distance between the two ships.

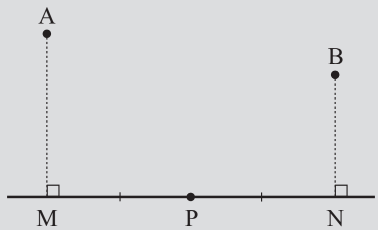
INVESTIGATION — SHORTEST DISTANCE

A and B are two farm houses which are 4 km and 3 km away from a water pipeline. M and N are the nearest points (on the pipeline) to A and B respectively, and MN = 6 km. The cost of running a spur pipeline across country from the main pipe line is \$3000 per km and the cost of a pump is \$8000. Your task is to determine the most economic way of pumping the water from the pipeline to A and B. Should you use two pumps (located at M and N) or use one pump located somewhere between M and N knowing that one pump would be satisfactory to pump sufficient water to meet the needs of both farm houses?



What to do:

1 Find the total cost of the pumps and pipelines if two pumps are used (one at M and the other at N).



2 Suppose one pump is used and it is located at P, the midpoint of MN.

  a Find AP and PB to the nearest metre.

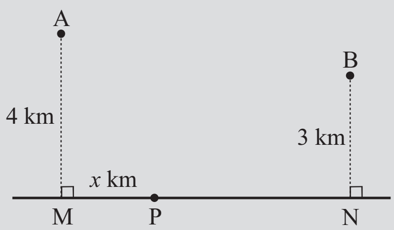
  b Find the total cost of the pipeline and pump in this case.

3 Now suppose P is x km from M.

  a Find distance AP in terms of x.

  b Find distance BP in terms of x.

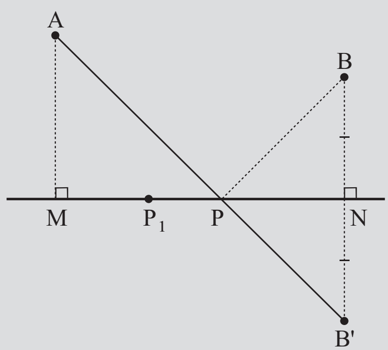
  c Show that AP + BP is given by $\sqrt{x^2+16}+\sqrt{x^2-12x+45}$ km.



4 Use a **graphics calculator** to find the smallest value of AP + BP using the formula in **3**.

5 From your answer in **4** calculate the total cost of the pipeline and pump for the shortest distance AP + BP.

6



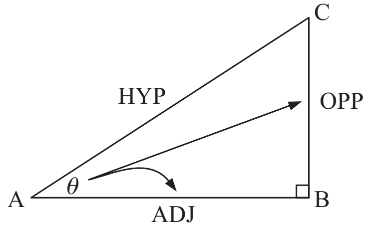
To locate P geometrically we reflect B in the mirror line MN. Its image is B′. We then join AB′. P is located where AB′ meets MN.

Show that the above statement is correct. (**Hint:** AP + PB = AP + PB′ and compare APB′ with AP_1B' for any other point P_1 on MN.)

C RIGHT ANGLED TRIANGLE TRIGONOMETRY

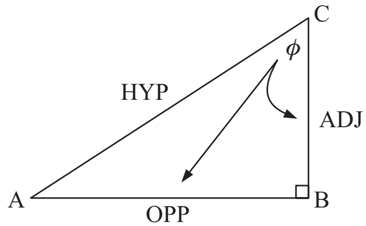
LABELLING RIGHT ANGLED TRIANGLES

For the given right angled triangle, the **hypotenuse (HYP)** is the side which is opposite the right angle and is the longest side of the triangle.



For the angle marked θ:

- BC is the side **opposite (OPP)** angle θ
- AB is the side **adjacent (ADJ)** angle θ.

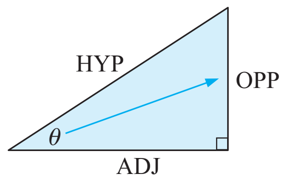


Notice that, for the angle marked ϕ:

- AB is the side **opposite (OPP)** angle ϕ
- BC is the side **adjacent (ADJ)** angle ϕ.

THE THREE BASIC TRIGONOMETRIC RATIOS

By definition, the three basic trigonometric ratios are sine, cosine and tangent where



$$\sin\theta = \frac{\text{OPP}}{\text{HYP}}, \qquad \cos\theta = \frac{\text{ADJ}}{\text{HYP}}, \qquad \tan\theta = \frac{\text{OPP}}{\text{ADJ}}.$$

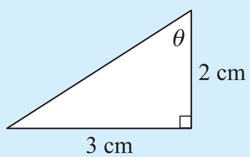
$\sin\theta$, $\cos\theta$ and $\tan\theta$ are abbreviations for sine θ, cosine θ and tangent θ.

The three formulae above are called the **trigonometric ratios** and are the tools we use for finding sides and angles of right angled triangles.

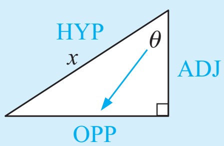
However, before doing this we will calculate the trigonometric ratios in right angled triangles where we know two of the sides.

Example 4

Given



find without using a calculator $\sin\theta$, $\cos\theta$ and $\tan\theta$.



If the hypotenuse is x cm long

$x^2 = 2^2 + 3^2$ {Pythagoras}

$\therefore\ x^2 = 13$

$\therefore\ x = \pm\sqrt{13}$

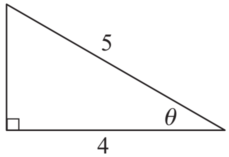
$\therefore\ x = \sqrt{13}$ {as $x > 0$}

So, $\sin\theta = \frac{3}{\sqrt{13}}$ $\qquad \cos\theta = \frac{2}{\sqrt{13}}$ $\qquad \tan\theta = \frac{3}{2}$.

EXERCISE C

1 For the following triangles, find the length of the third side and hence find $\sin\theta$, $\cos\theta$ and $\tan\theta$:

a



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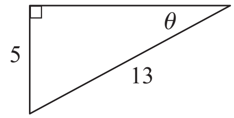

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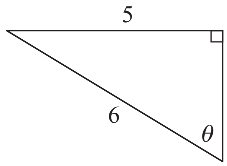
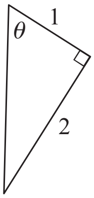
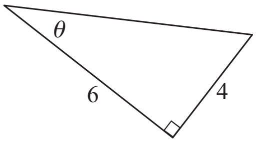


d



e



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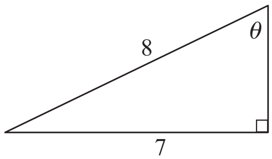


Example 5

If θ is an acute angle and $\sin\theta = \frac{1}{3}$, find $\cos\theta$ and $\tan\theta$ without using a calculator.

We draw a right angled triangle with OPP $= 1$ unit
HYP $= 3$ units

Now $x^2 + 1^2 = 3^2$ {Pythagoras}

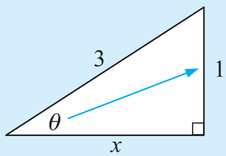
$\therefore\ x^2 + 1 = 9$

$\therefore\ x^2 = 8$

$\therefore\ x = \pm\sqrt{8}$

so, $x = \sqrt{8}$ as $x > 0$.

$\therefore\ \cos\theta = \frac{\sqrt{8}}{3}$ and $\tan\theta = \frac{1}{\sqrt{8}}$.



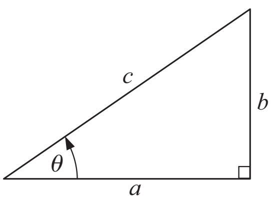
Note: Generally, if $\sin\theta = \frac{1}{3}$ we cannot say that OPP $= 1$ and HYP $= 3$ as it could be that OPP $= 2$ and HYP $= 6$. However, using the simplest ratio produces the required result.

2 **a** If θ is an acute angle and $\cos\theta = \frac{1}{2}$, find $\sin\theta$ and $\tan\theta$.

b If α is an acute angle and $\sin\alpha = \frac{2}{3}$, find $\cos\alpha$ and $\tan\alpha$.

c If β is an acute angle and $\tan\beta = \frac{4}{3}$, find $\sin\beta$ and $\cos\beta$.

3

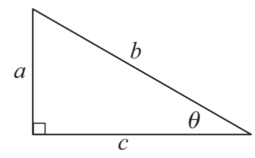


a For the triangle given, write down expressions for $\sin\theta$, $\cos\theta$ and $\tan\theta$.

b Write $\dfrac{\sin\theta}{\cos\theta}$ in terms of a, b and c

and hence show that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$.

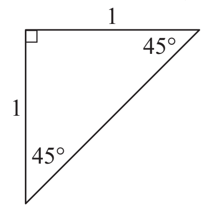
4 The remaining angle of the illustrated triangle is $90-\theta$, which is the complement of θ.



Recall: Two angles are *complementary* if their sum is 90^o.

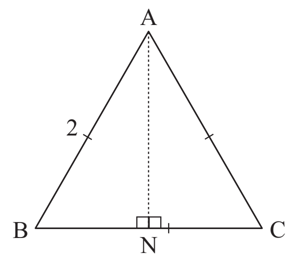
a Find:
i $\sin\theta$ **ii** $\cos\theta$ **iii** $\sin(90-\theta)$ **iv** $\cos(90-\theta)$

b Use your results of **a** to complete the following statements:
i *The sine of an angle is the cosine of its ......*
ii *The cosine of an angle is the sine of its ......*



5 **a** Find the length of the remaining side.
b Find $\sin 45^o$, $\cos 45^o$, $\tan 45^o$ using the figure.
c Use your calculator to check your answers.

6 Triangle ABC is equilateral. AN is the altitude to BC.
a State the measures of angles ABN and BAN.
b Find the length of BN and AN.
c Hence, without using a calculator, find:
i $\sin 60^o$, $\cos 60^o$ and $\tan 60^o$
ii $\sin 30^o$, $\cos 30^o$ and $\tan 30^o$.



We can summarise the ratios for special angles in table form.

Try to learn them.

Summary

θ (degrees)	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

D FINDING SIDES AND ANGLES

NOTE ON CALCULATOR USE

Before commencing calculations make sure that you check that the MODE is set on **degrees**. In this chapter all problem solving is in degrees.

SIDE FINDING

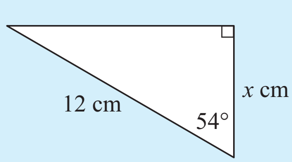
In a right angled triangle, if we wish to find the **length of a side** we first need to know:

- one angle and
- one other side.

Example 6

Find, correct to 3 significant figures, the value of x in:

a



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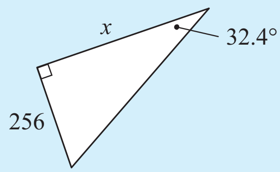
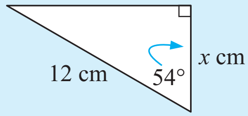


a



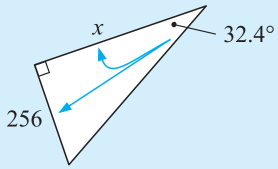
For the 54^o angle, HYP $= 12$, ADJ $= x$.

$$\text{So,} \quad \cos 54^o = \frac{x}{12}$$

$$\therefore \quad 12\cos 54^o = x$$

$$\therefore \quad x \doteq 7.05$$

b For the 32.4^o angle, OPP $= 12$, ADJ $= x$



$$\text{So,} \quad \tan 32.4^o = \frac{256}{x}$$

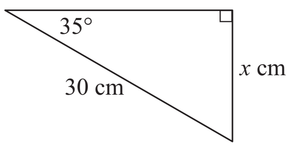
$$\therefore \quad x = \frac{256}{\tan 32.4^o} \doteq 403.4$$

$$\therefore \quad x \doteq 403$$

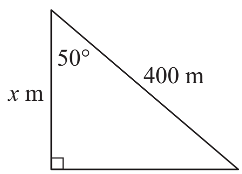
EXERCISE D

1 Find, correct to 3 significant figures, the value of the unknown in each of the following:

a



b



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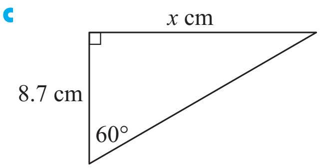
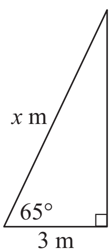
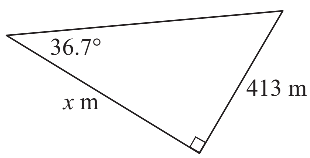


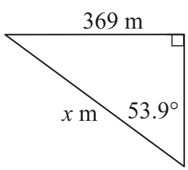
d



e



f



ANGLE FINDING

In a right angled triangle, if we wish to find the size of an acute angle we need to know the **lengths of two sides**.

Reminder:

- if $\sin\theta = \dfrac{a}{b}$ then $\theta = \arcsin\left(\dfrac{a}{b}\right)$ which reads 'the angle with a sine of $\dfrac{a}{b}$'
- if $\cos\theta = \dfrac{a}{b}$ then $\theta = \arccos\left(\dfrac{a}{b}\right)$ which reads 'the angle with a cosine of $\dfrac{a}{b}$'
- if $\tan\theta = \dfrac{a}{b}$ then $\theta = \arctan\left(\dfrac{a}{b}\right)$ which reads 'the angle with a tangent of $\dfrac{a}{b}$'.

When using your calculator to find, for example $\arcsin(\frac{3}{4})$ or $\sin^{-1}(\frac{3}{4})$

Use **2nd**, **INV** or **SHIFT** **before** **sin**.

depending on what calculator you are using

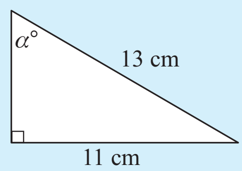
2 Use your calculator to find the acute angle θ in degrees to 3 significant figures:

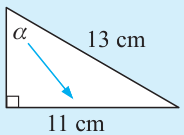
a $\sin\theta = 0.9364$ **b** $\cos\theta = 0.2381$ **c** $\tan\theta = 1.7321$ **d** $\cos\theta = \frac{2}{7}$

e $\sin\theta = \frac{1}{3}$ **f** $\tan\theta = \frac{14}{3}$ **g** $\sin\theta = \frac{\sqrt{3}}{11}$ **h** $\cos\theta = \frac{5}{\sqrt{37}}$

Example 7

Find α in degrees, correct to 3 significant figures:





For angle α, OPP = 11, HYP = 13.

So, $\sin\alpha = \frac{11}{13}$

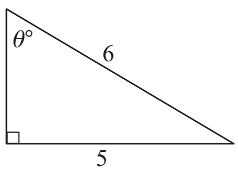
$\therefore \quad \alpha = \sin^{-1}(\frac{11}{13})$

$\therefore \quad \alpha \doteqdot 57.8$

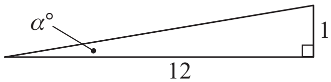
Steps: DEG mode, **2nd** **sin** 11 **÷** 13 **)** **ENTER**

3 Find, correct to 3 significant figures, the measure of the unknown angle in each of the following:

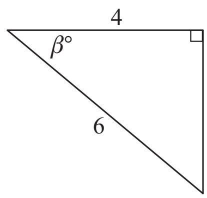
a



b

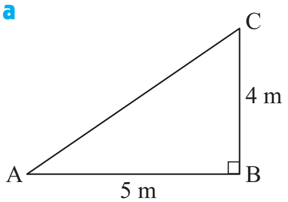


c

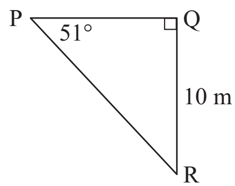


4 Solve the following triangles, i.e., find all unknown sides and angles:

a

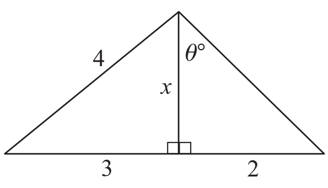


b

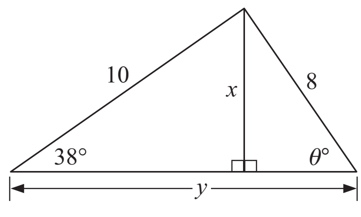


5 Find unknown sides and angles in the following figures:

a



b



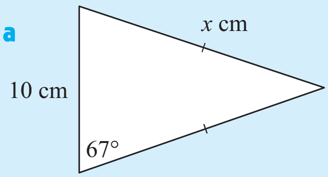
ISOSCELES TRIANGLES

To use trigonometry with isosceles triangles we invariably draw the **perpendicular** from the apex to the base. This altitude **bisects** the base.

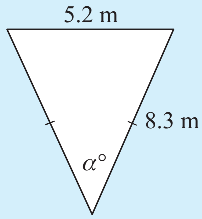
Example 8

Find, to 3 s.f. the unknowns in the following diagrams:

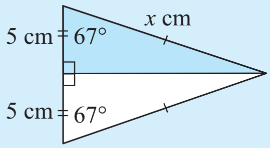
a



b



a

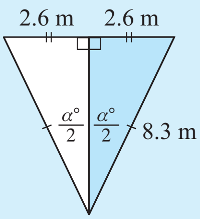


In the shaded right angled triangle

$$\cos 67^o = \frac{5}{x}$$

$$\therefore \quad x = \frac{5}{\cos 67^o} \doteqdot 12.8$$

b



In the shaded right angled triangle

$$\sin\left(\frac{\alpha}{2}\right) = \frac{2.6}{8.3}$$

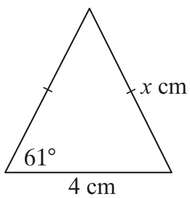
$$\therefore \quad \frac{\alpha}{2} = \sin^{-1}\left(\frac{2.6}{8.3}\right)$$

$$\therefore \quad \alpha = 2\sin^{-1}\left(\frac{2.6}{8.3}\right) \doteqdot 36.5$$

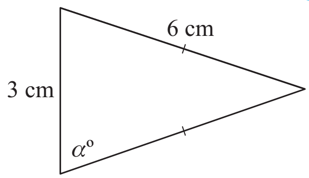
Steps: 2 [×] [2nd] [sin⁻¹] 2.6 [÷] 8.3 [)] [ENTER]

6 Find, correct to 3 significant figures, the unknowns in the following:

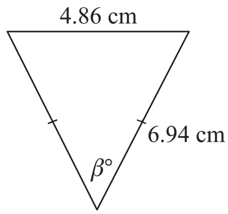
a



b

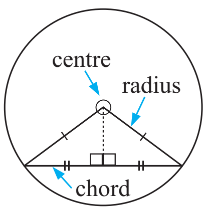


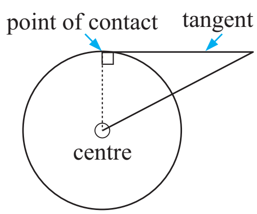
c

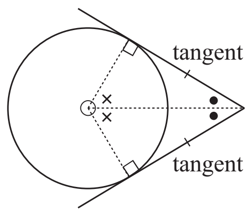


CHORDS AND TANGENTS

Right angled triangles occur in chord and tangent problems.

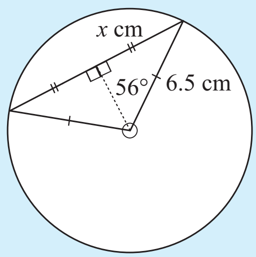






Example 9

A chord of a circle subtends an angle of 112^o at its centre. Find the length of the chord if the radius of the circle is 6.5 cm.



We complete an isosceles triangle and draw the line from the apex to the base.

For the 56^o angle, HYP $= 6.5$, OPP $= x$,

$$\sin 56^o = \frac{x}{6.5}$$

$$\therefore \quad 6.5 \times \sin 56^o = x$$

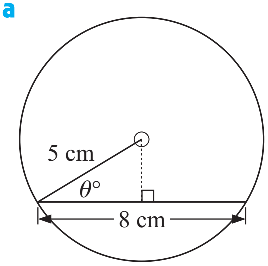
$$\therefore \quad x \doteqdot 5.389$$

$$\therefore \quad 2x \doteqdot 10.78$$

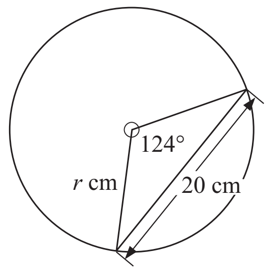
$\therefore$ chord is 10.8 cm long.

7 Find the value of the unknown in:

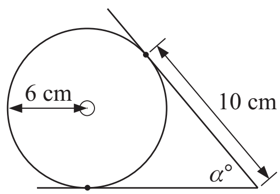
a



b



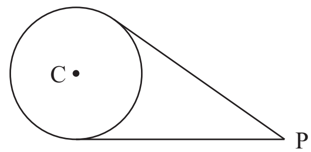
c



8 A chord of a circle subtends an angle of 89^o at its centre. Find the length of the chord given that the circle's diameter is 11.4 cm.

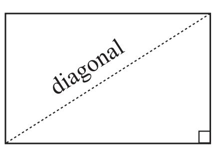
9 A chord of a circle is 13.2 cm long and the circle's radius is 9.4 cm. Find the angle subtended by the chord at the centre of the circle.

10 Point P is 10 cm from the centre of a circle of radius 4 cm. Tangents are drawn from P to the circle. Find the angle between the tangents.

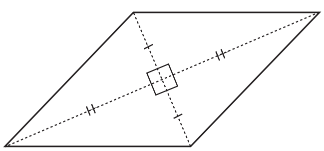


OTHER FIGURES

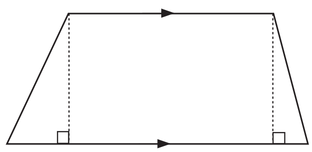
Sometimes right angled triangles can be found in other geometric figures such as rectangles, rhombi and trapezia.



rectangle



rhombus

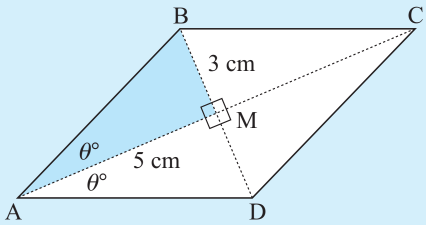


trapezium or trapezoid

Often right angled triangle trigonometry can be used in these figures if sufficient information is given.

Example 10

A rhombus has diagonals of length 10 cm and 6 cm respectively.
Find the smaller angle of the rhombus.



The diagonals bisect each other at right angles, so AM = 5 cm and BM = 3 cm.

In ΔABM, θ will be the smallest angle as it is opposite the shortest side.

$$\tan\theta = \tfrac{3}{5}$$
$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{3}{5}\right)$$
$$\text{i.e.,} \quad \theta \doteqdot 30.964$$

But the required angle is 2θ as the diagonals bisect the angles at each vertex,
$\therefore$ angle is 61.9^o.

11 A rectangle is 9.2 m by 3.8 m. What angle does its diagonal make with its longer side?

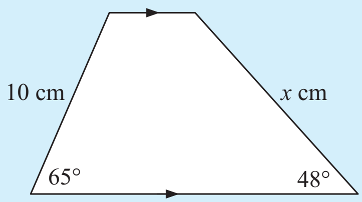
12 The diagonal and the longer side of a rectangle make an angle of 43.2^o. If the longer side is 12.6 cm, find the length of the shorter side.

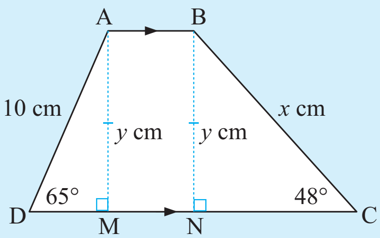
13 A rhombus has diagonals of length 12 cm and 7 cm respectively. Find the larger angle of the rhombus.

14 The smaller angle of a rhombus measures 21.8^o and the shorter diagonal is 13.8 cm. Find the lengths of the sides of the rhombus.

Example 11

Find x given:





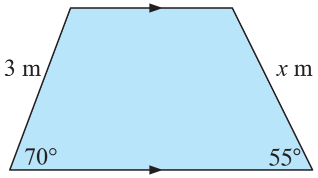
We draw perpendiculars AM and BN to DC creating right angled triangles and rectangle ABNM.

In ΔADM, $\sin 65^o = \dfrac{y}{10}$ and $\therefore \ y = 10\sin 65^o$

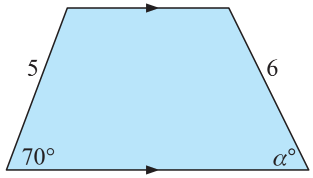
In ΔBCN, $\sin 48^o = \dfrac{y}{x} = \dfrac{10\sin 65^o}{x}$

$$\therefore \quad x = \frac{10\sin 65^o}{\sin 48^o} \doteqdot 12.2$$

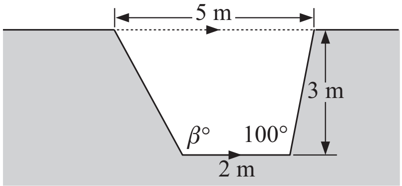
15 **a** Find the value of x in:



b Find the unknown angle in:



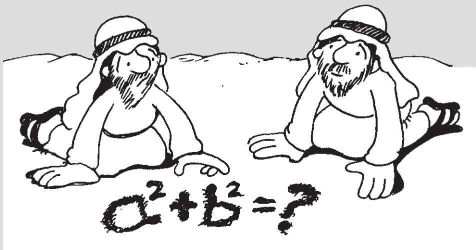
16 A stormwater drain is to have the shape as shown. Determine the angle the left hand side makes with the bottom of the drain.



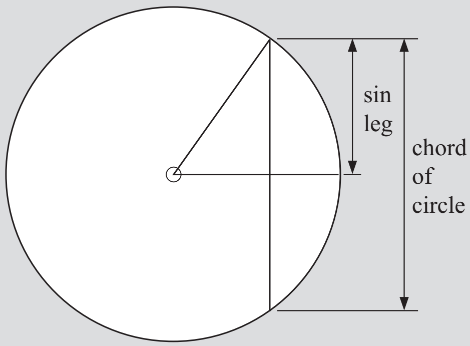
HISTORICAL NOTE

The origin of the term "sine" is quite fascinating. **Arbyabhata**, a Hindu mathematician who studied trigonometry in the 5th century AD, called the sine-leg of the circle diagram "ardha-jya" which means "half-chord".

This was eventually shortened to "jya". Arab scholars later translated Arbyabhata's work into Arabic and initially phonetically translated "jya" as "jiba" but since this meant nothing in Arabic they very shortly began writing the word as "jaib" which has the same letters but means "cove" or "bay".



Finally in 1150 an Italian, **Gerardo of Cremona**, translated this work into Latin and replaced "jaib" with "sinus" which means "bend" or "curve" but is commonly used in Latin to refer to a bay or gulf on a coastline. The term "sine" that we use today comes from this Latin word "sinus". The term "cosine" comes from the fact that the sine of an angle is equal to the cosine of its complement. In 1620, **Edmund Gunter** introduced the abbreviated "co sinus" for "complementary sine".

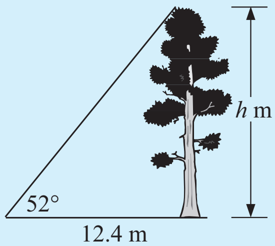


E PROBLEM SOLVING USING TRIGONOMETRY

Trigonometry is a very useful branch of mathematics. **Heights** and **distances** which are very difficult or even impossible to measure can often be found using **trigonometry**.

Example 12

Find the height of a tree which casts a shadow of 12.4 m when the sun makes an angle of 52^o to the horizon.



Let h m be the tree's height.

For the 52^o angle, OPP $= h$, ADJ $= 12.4$

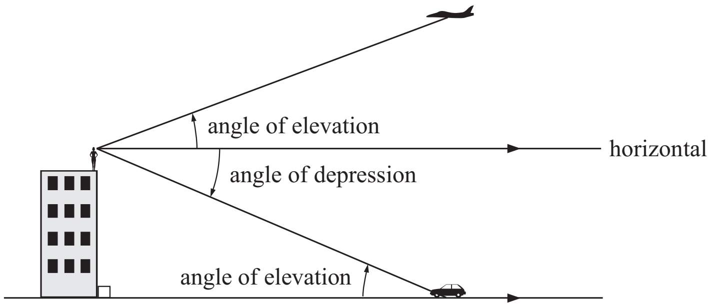
$$\therefore \quad \tan 52^o = \frac{h}{12.4}$$

$$\therefore \quad 12.4 \times \tan 52^o = h$$

$$\therefore \quad h \doteqdot 15.9$$

$\therefore$ tree is 15.9 m high.

Reminder:



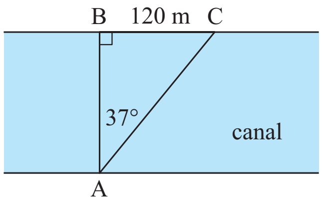
EXERCISE E

1 Find the height of a flagpole which casts a shadow of 9.32 m when the sun makes an angle of 63^o to the horizontal.

2 A hill is inclined at 18^o to the horizontal. If the base of the hill is at sea level find:

- **a** my height above sea level if I walk 1.2 km up the hill
- **b** how far I have walked up the hill if I am 500 metres above sea level.

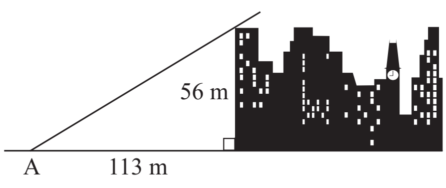
3



A surveyor standing at A notices two posts B and C on the opposite side of a canal. The posts are 120 m apart. If the angle of sight between the posts is 37^o, how wide is the canal?

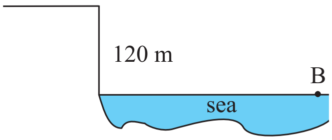
4 A train must climb a constant gradient of 5.5 m for every 200 m of track. Find the angle of incline.

5 Find the angle of elevation to the top of a 56 m high building from point A, which is 113 m from its base. What is the angle of depression from the top of the building to A?



6 The angle of depression from the top of a 120 m high vertical cliff to a boat B is 16^o.

Find how far the boat is from the base of the cliff.

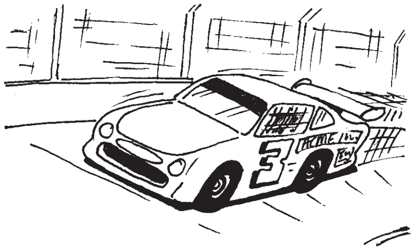


7 Sarah measures the angle of elevation to the top of a tree as 23.6^o from a point which is 250 m from its base. Her eye level, where the angle measurement was taken, is 1.5 m above the ground. Assuming the ground to be horizontal, find the height of the tree.

8 Kylie measures the angle of elevation from a point on level ground to the top of a building 120 metres high to be 32^o. She walks towards the building until the angle of elevation is 45^o. How far does she walk?

9 From a point A, 40 metres from the base of a building B, the angle of elevation to the top of the building C is 51^o, and to the top of the flagpole D on top of the building is 56^o. Find the height of the flagpole.

10 For a circular track of radius r metres, banked at θ degrees to the horizontal, the ideal velocity (the velocity that gives no tendency to sideslip) in metres per second is given by the formula: $v = \sqrt{gr\tan\theta}$, where $g = 9.8$ m/s^2.

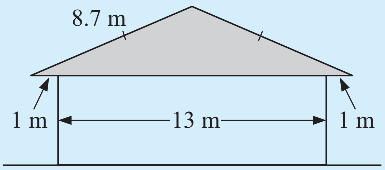


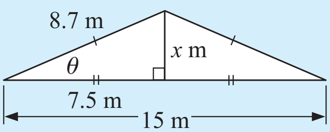
a What is the ideal velocity for a vehicle travelling on a circular track of radius 100 m, banked at an angle of 15^o?

b At what angle should a track of radius 200 m be banked, if it is designed for a vehicle travelling at 20 m/s?

Example 13

A builder designs a roof structure as illustrated. The pitch of the roof is the angle that the roof makes with the horizontal. Find the pitch of the roof.





Using the right angled triangle created from the isosceles triangle, for angle θ:

ADJ $= 7.5$, HYP $= 8.7$

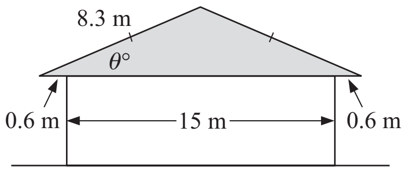
$$\therefore \quad \cos\theta = \frac{7.5}{8.7}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{7.5}{8.7}\right)$$

$$\therefore \quad \theta \doteqdot 30.450....$$

$\therefore$ the pitch is approximately $30\frac{1}{2}^o$.

11 Find θ, the pitch of the roof.



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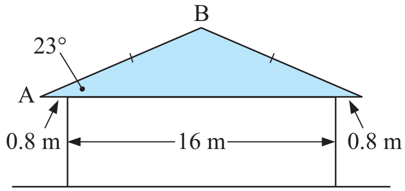
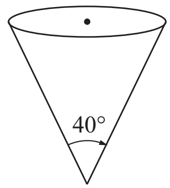


If the pitch of the given roof is 23^o, find the length of the timber beam AB.

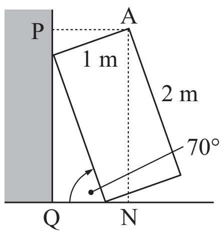
13 An open right-circular cone has a vertical angle measuring 40^o and a base radius of 30 cm.

Find the capacity of the cone in litres.

$$(V = \tfrac{1}{3}\pi r^2 h)$$



14

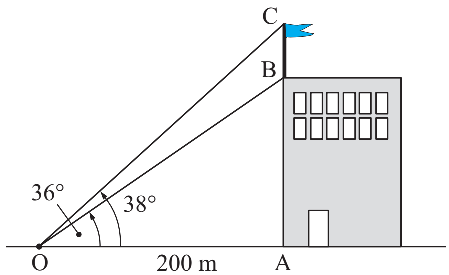


A refrigerator leans against a vertical wall making an angle of 70^o with the horizontal floor. How high is point A above the floor?

(**Hint:** AN = PQ)

15 From an observer O, the angles of elevation to the bottom and the top of a flagpole are 36^o and 38^o respectively. Find the height of the flagpole.

(**Hint:** Find AB and AC.)

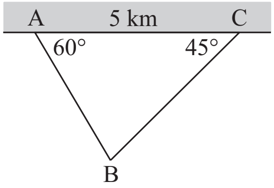


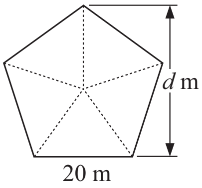
16 The angle of depression from the top of a 150 m high cliff to a boat at sea is 7^o. How much closer to the cliff must the boat move for the angle of depression to become 19^o?

17 A helicopter flies horizontally at 100 kmph. An observer notices that it took 20 seconds for the helicopter to fly from directly overhead to being at an angle of elevation of 60^o. Find the height of the helicopter above the ground.

18 A balloon travels horizontally at a distance h kilometres above the ground between two points A and B, which are two kilometres apart. From a point C on the ground, the angle of elevation of the balloon at A is 40^o and at B is 25^o. Assume that A, B and C are in the same plane and that A and B are on the same side of the observation point. Find the height h of the balloon.

19 AC is a straight shore line and B is a boat out at sea. Find the shortest distance from the boat to the shore if A and C are 5 km apart.



20 

A regular pentagonal garden plot is to be constructed. The sides are to be of length 20 m.

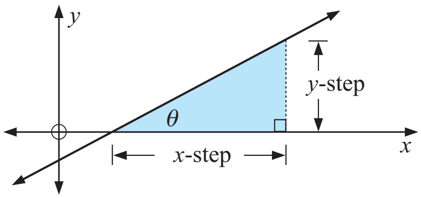
Find the width of land (d m) required for the plot.

F THE SLOPE OF A STRAIGHT LINE

If a straight line makes an angle of θ with the positive x-axis then its slope $m = \tan\theta$.

Proof:

$$\text{slope } m = \frac{y\text{-step}}{x\text{-step}}$$
$$= \tan\theta \qquad \{\text{in shaded } \Delta\}$$



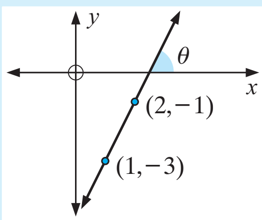
Example 14

Find the angle that the line through P(2, −1) and Q(1, −3) makes with the positive x-axis.

$$\tan\theta = m = \frac{(-3)-(-1)}{1-2}$$

$$\therefore \quad \tan\theta = \frac{-2}{-1} = 2$$

$$\therefore \quad \theta = \tan^{-1}(2) \doteqdot 63.4^o$$



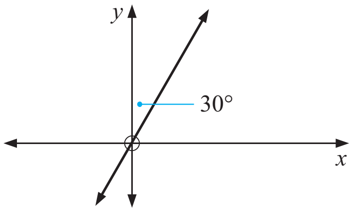
EXERCISE F

1 What angle does the line through:

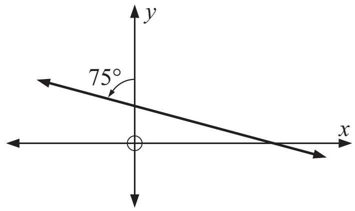
a A(2, 5) and B(−1, 4) make with the positive x-axis
b C(3, −2) and D(−1, −4) make with the positive x-axis
c E(−2, 1) and F(1, −5) make with the positive x-axis
d G(−7, 4) and H(−2, −1) make with the positive x-axis?

2 Find the slope of the following lines:

a

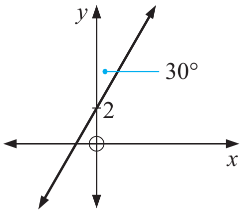


b

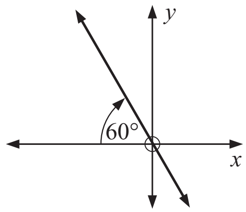


3 Find the equations of the following lines:

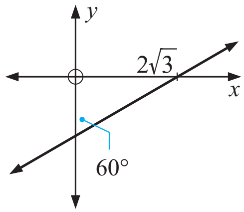
a



b

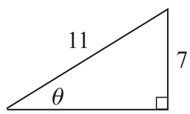


c

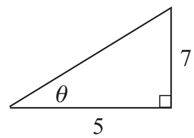


REVIEW SET A

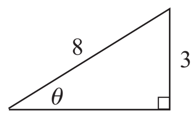
1 Find $\sin\theta$ for:



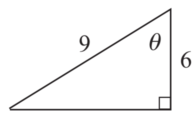
2 Find $\sin\theta$ and $\cos\theta$ for:



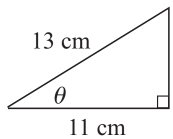
3 Find $\tan\theta$ for:



4 Find the unknown side and hence $\sin\theta$, $\cos\theta$, and $\tan\theta$:



5 Find the unknown side and hence find $\sin\theta$, $\cos\theta$ and $\tan\theta$:



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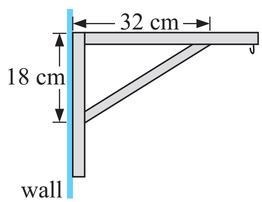


Metal brackets as shown alongside are attached to a wall so that hanging baskets may be hung from them. Using the dimensions given, find:

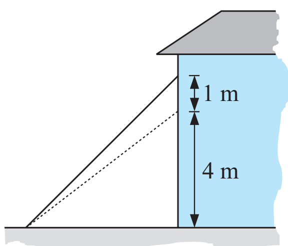
a the length of the diagonal support

b the angle the diagonal support makes with the wall.

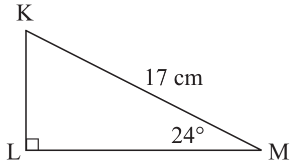
7 When an extension ladder rests against a wall it reaches 4 m up the wall. The ladder is extended a further 0.8 m without moving the foot of the ladder and it now rests against the wall 1 m further up. Find:

a the length of the extended ladder

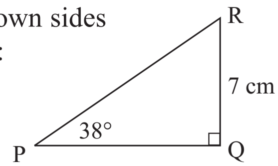
b the increase in the angle that the ladder makes with the ground now that the ladder is extended.



8 Find all unknown sides and angles.



9 Find the unknown sides and angles for:



10 Determine the height of a tree which casts a shadow of 13.7 m when the sun is at an angle of 28^o.

REVIEW SET B

1 A yacht sails 8.6 km due east and then 13.2 km south. Find the distance and bearing of the yacht from its starting point.

2 Find the angle of elevation to the top of a mountain 2300 m high from a point 5.6 km from its base.

3 An aeroplane flying at 10 000 m is at an angle of elevation of 36^o. If two minutes later, the angle of elevation is 21^o, determine the speed of the plane.

4 Find the acute angle θ, if:

a $\sin\theta = 0.8147$ **b** $\cos\theta = 0.0917$ **c** $\tan\theta = 5.23$

5 Find the acute angle θ, if:

a $\sin\theta = \frac{\sqrt{11}}{5}$ **b** $\cos\theta = \frac{5}{7}$ **c** $\tan\theta = 0.7452$

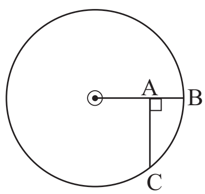
6 A chord of a circle subtends an angle of 114^o at its centre. Find the radius of the circle given that the length of the chord is 10.4 cm.

7 A tangent from a point P is drawn to a circle of radius 6.4 cm. Find the angle between the tangent and the line joining P to the centre of the circle if the tangent has length 13.6 cm.

8 In the given figure AB = 1 cm and AC = 3 cm. Find:

a the radius of the circle

b the angle subtended by chord BC at the centre of the circle.



9 The larger angle of a rhombus measures 114^o and the longer diagonal is 16.4 cm. Find the lengths of the sides of the rhombus.

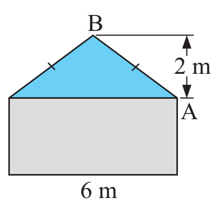
10 A flagpole 19.6 m high is supported by 3 wires which meet the ground at an angle of 56^o. Determine the total length of the three wires.

REVIEW SET C

1 In the illustrated roof structure:

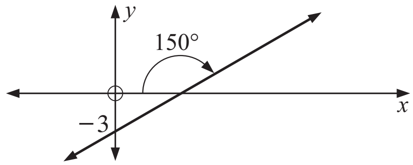
a how long is the timber beam AB

b at what angle is the beam inclined to the horizontal?



2 What angle does the line through P(7, -4) and Q(1, 6) make with the positive x-axis?

3 Find the equation of this line:



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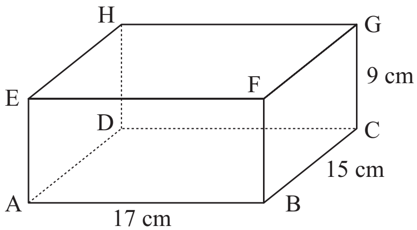


a Sketch triangle DHE showing which angle is the right angle.

b Find the measure of angle HDE.

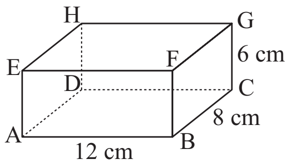
c Sketch triangle ACG showing which angle is the right angle.

d Find the measure of angle AGC.

5 Find the angle that:

a AG makes with BG

b DF makes with DB.



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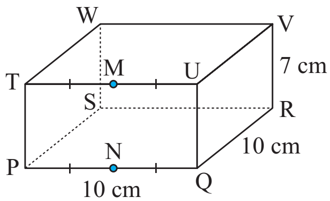


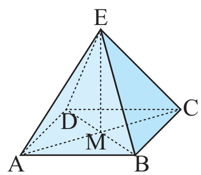
M and N are the midpoints of TU and PQ respectively.

a Draw a sketch of triangle RMN showing which angle is the right angle.

b Find the length of RN.

c Find the measure of angle RMN.

7 ABCD is a square-based pyramid. E, the apex of the pyramid is vertically above M, the point of intersection of AC and BD. If an Egyptian Pharoah wished to build a square-based pyramid with all edges 200 m, find:



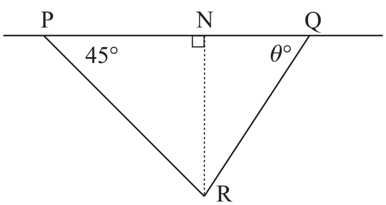
a how high (to the nearest metre) the pyramid would reach above the desert sands

b the measure of the angle between a slant edge and a base diagonal.

8 An isosceles triangle is drawn with base angles 32^o and base 24 cm. Find the base angles of an isosceles triangle with the same base but with double the area.

9 PQ is a straight shore line and R is a boat out at sea. Show that if P and Q are 5 km apart the shortest distance from the boat to shore is given by

$$\text{RN} = \frac{5\tan\theta}{1+\tan\theta}.$$



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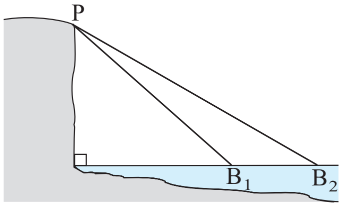


From the top of a cliff 200 m above sea level the angles of depression to two fishing boats are 6.7^o and 8.2^o respectively. How far apart are the boats?

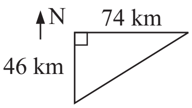
Background knowledge in Trigonometry

EXERCISE A

1 **a** $x = 0.663$ **b** $x = 4.34$ **c** $x = 2.23$

2 **a** 4.54 m **b** 4.17 m **3** 237 m

4 **a**



b 87.1 km

EXERCISE B

1 50.3 m **2** 17.3 cm **3** 8.60 m **4** 53.4 m

5 44.4 km

EXERCISE C

1 **a** 3, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$

b 12, $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{5}{12}$

c $\sqrt{11}$, $\sin\theta = \frac{5}{6}$, $\cos\theta = \frac{\sqrt{11}}{6}$, $\tan\theta = \frac{5}{\sqrt{11}}$

d $\sqrt{5}$, $\sin\theta = \frac{2}{\sqrt{5}}$, $\cos\theta = \frac{1}{\sqrt{5}}$, $\tan\theta = 2$

e $\sqrt{52}$, $\sin\theta = \frac{4}{\sqrt{52}}$, $\cos\theta = \frac{6}{\sqrt{52}}$, $\tan\theta = \frac{2}{3}$

f $\sqrt{15}$, $\sin\theta = \frac{7}{8}$, $\cos\theta = \frac{\sqrt{15}}{8}$, $\tan\theta = \frac{7}{\sqrt{15}}$

2 **a** $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = \sqrt{3}$ **b** $\cos\alpha = \frac{\sqrt{5}}{3}$, $\tan\alpha = \frac{2}{\sqrt{5}}$

c $\sin\beta = \frac{4}{5}$, $\cos\beta = \frac{3}{5}$

3 **a** $\sin\theta = \dfrac{b}{c}$, $\cos\theta = \dfrac{a}{c}$, $\tan\theta = \dfrac{b}{a}$

4 **a** **i** $\dfrac{a}{b}$ **ii** $\dfrac{c}{b}$ **iii** $\dfrac{c}{b}$ **iv** $\dfrac{a}{b}$ **b** **i** complement **ii** complement

5 **a** $\sqrt{2}$ **b** $\sin 45^o = \frac{1}{\sqrt{2}}$, $\cos 45^o = \frac{1}{\sqrt{2}}$, $\tan 45^o = 1$

6 **a** $\measuredangle ABN = 60^o$, $\measuredangle BAN = 30^o$ **b** BN = 1, AN = $\sqrt{3}$

c **i** $\sin 60^o = \frac{\sqrt{3}}{2}$, $\cos 60^o = \frac{1}{2}$, $\tan 60^o = \sqrt{3}$

ii $\sin 30^o = \frac{1}{2}$, $\cos 30^o = \frac{\sqrt{3}}{2}$, $\tan 30^o = \frac{1}{\sqrt{3}}$

EXERCISE D

1 **a** $x = 17.2$ **b** $x = 257$ **c** $x = 15.1$ **d** $x = 7.10$
e $x = 554$ **f** $x = 457$

2 **a** $\theta = 69.5^o$ **b** $\theta = 76.2^o$ **c** $\theta = 60.0^o$ **d** $\theta = 73.4^o$
e $\theta = 19.5^o$ **f** $\theta = 77.9^o$ **g** $\theta = 9.06^o$ **h** $\theta = 34.7^o$

3 **a** $\theta = 56.4$ **b** $\alpha = 4.76$ **c** $\beta = 48.2$

4 **a** AC = 6.40 m, $\measuredangle A = 38.7^o$, $\measuredangle C = 51.3^o$
b $\measuredangle R = 39^o$, PQ = 8.10 m, PR = 12.9 m

5 **a** $x = 2.65$, $\theta = 37.1$ **b** $x = 6.16$, $\theta = 50.3$, $y = 13.0$

6 **a** $x = 4.13$ **b** $\alpha = 75.5$ **c** $\beta = 41.0$

7 **a** $\theta = 36.9$ **b** $r = 11.3$ **c** $\alpha = 61.9$ **8** 7.99 cm

9 89.2^o **10** 47.2^o **11** 22.4^o **12** 11.8 cm **13** 119^o

14 36.5 cm **15** **a** $x = 3.44$ **b** $\alpha = 51.5$ **16** 129^o

EXERCISE E

1 18.3 m **2** **a** 371 m **b** 1.62 km **3** 159 m **4** 1.58^o

5 angle of elevn. $= 26.4^o$, angle of depn. $= 26.4^o$

6 418.5 m **7** 111 m **8** 72.0 m **9** 9.91 m

10 **a** 16.2 m/s **b** 11.5^o **11** $\theta = 12.6$ **12** 9.56 m

13 77.7 litres **14** 2.22 m **15** 10.95 m **16** 786 m

17 962 m **18** 2.10 km **19** 3.17 km **20** 30.8 m

EXERCISE F

1 **a** $\doteqdot 18.4^o$ **b** $\doteqdot 26.6^o$ **c** $\doteqdot 116.6^o$ **d** 135^o

2 **a** $\tan 60^o = \sqrt{3}$ **b** $\tan 165^o \doteqdot -0.268$

3 **a** $y = \sqrt{3}x + 2$ **b** $y = -\sqrt{3}x$ **c** $y = \frac{1}{\sqrt{3}}x - 2$

REVIEW SET A

1 $\frac{7}{11}$ **2** $\sin\theta = \frac{7}{\sqrt{74}}$, $\cos\theta = \frac{5}{\sqrt{74}}$ **3** $\frac{3}{\sqrt{55}}$

4 unknown side $= \sqrt{45} \doteqdot 6.71$, $\sin\theta = \frac{\sqrt{45}}{9}$,
$\cos\theta = \frac{2}{3}$, $\tan\theta = \frac{\sqrt{45}}{6}$

5 unknown side $= \sqrt{48} \doteqdot 6.93$ cm, $\sin\theta = \frac{\sqrt{48}}{13}$,
$\cos\theta = \frac{11}{13}$, $\tan\theta = \frac{\sqrt{48}}{11}$

6 **a** 36.7 cm **b** 60.6^o **7** **a** 6.03 m **b** 6.13^o

8 $\measuredangle K = 66^o$, KL = 6.91 cm, LM = 15.5 cm

9 $\measuredangle R = 52^o$, PQ = 8.96 cm, PR = 11.4 cm **10** 7.28 m

REVIEW SET B

1 15.8 km on a bearing of 147^o **2** 22.3^o

3 369 kmph **4** **a** $\theta = 54.6^o$ **b** $\theta = 84.7^o$ **c** $\theta = 79.2^o$

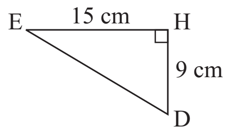
5 **a** $\theta = 41.6^o$ **b** $\theta = 44.4^o$ **c** $\theta = 36.7^o$ **6** 6.2 cm

7 25.2^o **8** **a** 5 cm **b** 36.9^o **9** 9.78 cm **10** 70.9 m

REVIEW SET C

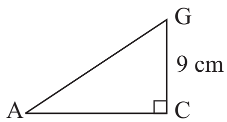
1 **a** 3.61 m **b** 33.7^o **2** $\doteqdot 121^o$ **3** $y = \frac{1}{\sqrt{3}}x - 3$

4 **a**



b 59.0^o

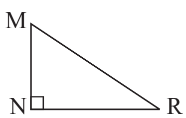
c



d 68.3^o

5 **a** 50.2^o **b** 22.6^o

6 **a**



b 11.2 cm
c 57.9^o

7 **a** 141 m **b** 45^o **8** $\doteqdot 51.3^o$ **10** $\doteqdot 315$ m