1. Given that $\dfrac{z}{z+2} = 2 - \text{i}$, $z \in \mathbb{C}$, find z in the form $a + \text{i}b$.

(Total 4 marks)

2. The complex numbers $z_1 = 2 - 2\text{i}$ and $z_2 = 1 - \text{i}\sqrt{3}$ are represented by the points A and B respectively on an Argand diagram. Given that O is the origin,

(a) find AB, giving your answer in the form $a\sqrt{b - \sqrt{3}}$, where $a, b \in \mathbb{Z}^+$;

(3)

(b) calculate $\text{A}\hat{\text{O}}\text{B}$ in terms of π.

(3)
(Total 6 marks)

3. Given that $z = \cos\theta + \text{i} \sin\theta$ show that

(a) $\text{Im}\left(z^n + \dfrac{1}{z^n}\right) = 0, n \in \mathbb{Z}^+$;

(2)

(b) $\text{Re}\left(\dfrac{z-1}{z+1}\right) = 0, z \neq -1$.

(5)
(Total 7 marks)

4. Consider the complex number $\omega = \dfrac{z+\text{i}}{z+2}$, where $z = x + \text{i}y$ and $\text{i} = \sqrt{-1}$.

(a) If $\omega = \text{i}$, determine z in the form $z = r \text{ cis } \theta$.

(6)

(b) Prove that $\omega = \dfrac{(x^2 + 2x + y^2 + y) + \mathrm{i}(x + 2y + 2)}{(x+2)^2 + y^2}$.

(3)

(c) **Hence** show that when Re(ω) = 1 the points (x, y) lie on a straight line, l_1, and write down its gradient.

(4)

(d) Given arg (z) = arg(ω) = $\dfrac{\pi}{4}$, find $|z|$.

(6)
(Total 19 marks)

5. Consider the complex numbers $z = 1 + 2\mathrm{i}$ and $w = 2 + a\mathrm{i}$, where $a \in \mathbb{R}$.

Find a when

(a) $|w| = 2|z|$;

(3)

(b) Re (zw) = 2 Im(zw).

(3)
(Total 6 marks)

6. If z is a non-zero complex number, we define $L(z)$ by the equation

$$L(z) = \ln|z| + \mathrm{i}\arg(z),\ 0 \le \arg(z) < 2\pi.$$

(a) Show that when z is a positive real number, $L(z) = \ln z$.

(2)

(b) Use the equation to calculate

(i) $L(-1)$;

(ii) $L(1 - \mathrm{i})$;

(iii) $L(-1 + \quad \mathrm{i})$.

(5)

(c) Hence show that the property $L(z_1z_2) = L(z_1) + L(z_2)$ does not hold for all values of z_1 and z_2.

(2)
(Total 9 marks)

7. Find, in its simplest form, the argument of $(\sin\theta + \mathrm{i}\,(1 - \cos\theta))^2$ where θ is an acute angle.

(Total 7 marks)

8. Consider $w = \dfrac{z}{z^2+1}$ where $z = x + \mathrm{i}y$, $y \neq 0$ and $z^2 + 1 \neq 0$.

Given that Im $w = 0$, show that $|z| = 1$.

(Total 7 marks)

9. (a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \mathrm{i}$.

(6)

(b) Draw these roots on an Argand diagram.

(2)

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\dfrac{z_2}{z_1}$ in the form $a + \mathrm{i}b$.

(4)
(Total 12 marks)

10. Given that $(a + b\mathrm{i})^2 = 3 + 4\mathrm{i}$ obtain a pair of simultaneous equations involving a and b. Hence find the two square roots of $3 + 4\mathrm{i}$.

(Total 7 marks)

11. Given that $|z| = \sqrt{10}$, solve the equation $5z + \dfrac{10}{z^*} = 6 - 18\mathrm{i}$, where z^* is the conjugate of z.

(Total 7 marks)

12. Solve the simultaneous equations

$$\mathrm{i}z_1 + 2z_2 = 3$$
$$z_1 + (1 - \mathrm{i})z_2 = 4$$

giving z_1 and z_2 in the form $x + \mathrm{i}y$, where x and y are real.

(Total 9 marks)

13. Find b where $\frac{2+b\mathrm{i}}{1-b\mathrm{i}} = \frac{7}{10} + \frac{9}{10}\mathrm{i}$.

(Total 6 marks)

14. Given that $z = (b + \mathrm{i})^2$, where b is real and positive, find the value of b when $\arg z = 60°$.

(Total 6 marks)

15. Consider the complex geometric series $\mathrm{e}^{\mathrm{i}\theta} + \frac{1}{2}\mathrm{e}^{2\mathrm{i}\theta} + \frac{1}{4}\mathrm{e}^{3\mathrm{i}\theta} + \ldots.$

(a) Find an expression for z, the common ratio of this series.

(2)

(b) Show that $|z| < 1$.

(2)

(c) Write down an expression for the sum to infinity of this series.

(2)

(d) (i) Express your answer to part (c) in terms of $\sin\theta$ and $\cos\theta$.

(ii) Hence show that

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \frac{4\cos\theta - 2}{5 - 4\cos\theta}.$$

(10)
(Total 16 marks)

16. The roots of the equation $z^2 + 2z + 4 = 0$ are denoted by α and β?

(a) Find α and β in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

(6)

(b) Given that α lies in the second quadrant of the Argand diagram, mark α and β on an Argand diagram.

(2)

(c) Use the principle of mathematical induction to prove De Moivre's theorem, which states that $\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n$ for $n \in \mathbb{Z}^+$.

(8)

(d) Using De Moivre's theorem find $\dfrac{\alpha^3}{\beta^2}$ in the form $a + \mathrm{i}b$.

(4)

(e) Using De Moivre's theorem or otherwise, show that $\alpha^3 = \beta^3$.

(3)

(f) Find the exact value of $\alpha\beta^* + \beta\alpha^*$ where α^* is the conjugate of α and β^* is the conjugate of β.

(5)

(g) Find the set of values of n for which α^n is real.

(3)
(Total 31 marks)