1. (a) (i) $f'(x) = \dfrac{x\frac{1}{x} - \ln x}{x^2}$ M1A1

$= \dfrac{1 - \ln x}{x^2}$

so $f'(x) = 0$ when $\ln x = 1$, i.e. $x = \mathrm{e}$ A1

(ii) $f'(x) > 0$ when $x < \mathrm{e}$ and $f'(x) < 0$ when $x > \mathrm{e}$ R1

hence local maximum AG

Note: Accept argument using correct second derivative.

(iii) $y \le \dfrac{1}{\mathrm{e}}$ A1

(b) $f''(x) = \dfrac{x^2\frac{-1}{x} - (1 - \ln x)2x}{x^4}$ M1

$= \dfrac{-x - 2x + 2x\ln x}{x^4}$

$= \dfrac{-3 + 2\ln x}{x^3}$ A1

Note: May be seen in part (a).

$f''(x) = 0$

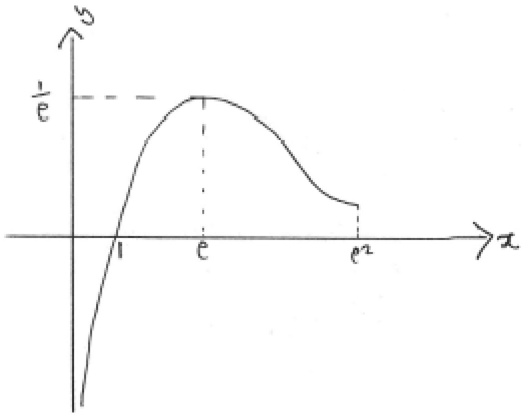
$-3 + 2\ln x = 0$ M1

$x = \mathrm{e}^{\frac{3}{2}}$

since $f''(x) < 0$ when $x < \mathrm{e}^{\frac{3}{2}}$ and $f''(x) > 0$ when $x > \mathrm{e}^{\frac{3}{2}}$ R1

then point of inflexion $\left(\mathrm{e}^{\frac{3}{2}}, \dfrac{3}{2\mathrm{e}^{\frac{3}{2}}}\right)$ A1

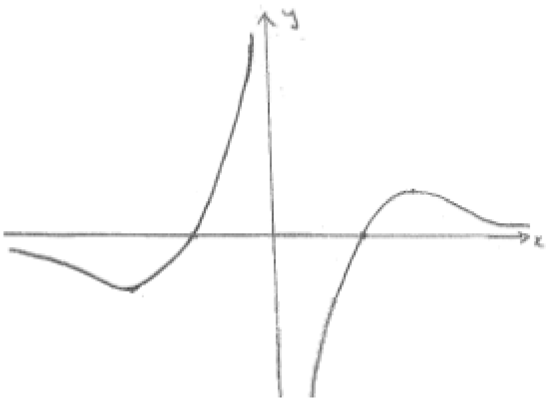
(c)



A1A1A1

Note: Award A1 for the maximum and intercept, A1 for a vertical asymptote and A1 for shape (including turning concave up).

(d) (i)

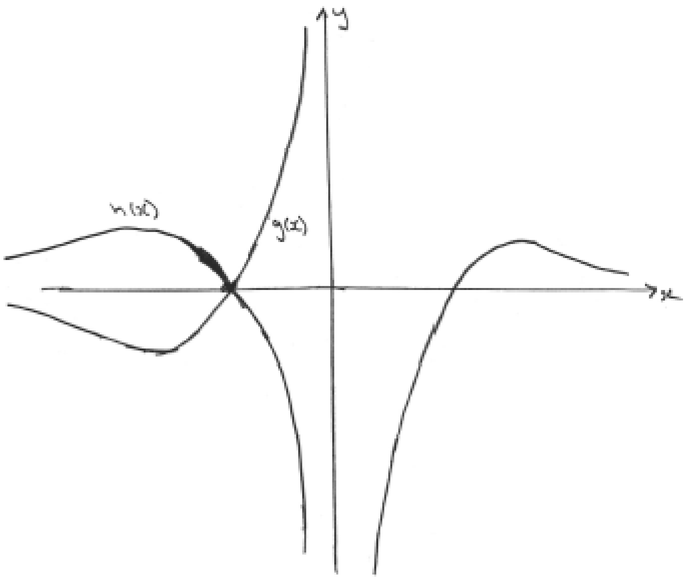


A1A1

Note: Award A1 for each correct branch.

(ii) all real values A1

(iii)



(M1)(A1)

Note: Award (M1)(A1) for sketching the graph of h, ignoring any graph of g.

$-e^2 < x < -1$ (accept $x < -1$) A1

[19]

2. (a) **METHOD 1**

$f'(x) = q - 2x = 0$ M1

$f'(3) = q - 6 = 0$

$q = 6$ A1

$f(3) = p + 18 - 9 = 5$ M1

$p = -4$ A1

METHOD 2

$f(x) = -(x-3)^2 + 5$ M1A1

$= -x^2 + 6x - 4$

$q = 6, p = -4$ A1A1

(b) $g(x) = -4 + 6(x-3) - (x-3)^2$ $(= -31 + 12x - x^2)$ M1A1

Note: Accept any alternative form that is correct.
Award M1A0 for a substitution of $(x + 3)$.

[6]

3. (a) $f'(x) = 3x^2 - 6x - 9\ (= 0)$ (M1)

$(x + 1)(x - 3) = 0$

$x = -1;\ x = 3$

(max)$(-1, 15)$; (min)$(3, -17)$ A1A1

Note: The coordinates need not be explicitly stated but the values need to be seen.

$y = -8x + 7$ A1 N2

(b) $f''(x) = 6x - 6 = 0 \Rightarrow$ inflexion $(1, -1)$ A1

which lies on $y = -8x + 7$ R1AG

[6]

4. (a) $x^2 + 5x + 4 = 0 \Rightarrow x = -1$ or $x = -4$ (M1)

so vertical asymptotes are $x = -1$ and $x = -4$ A1

as $x \to \infty$ then $y \to 1$ so horizontal asymptote is $y = 1$ (M1)A1

(b) $x^2 - 5x + 4 = 0 \Rightarrow x = 1$ or $x = 4$ A1

$x = 0 \Rightarrow y = 1$

so intercepts are $(1, 0)$, $(4, 0)$ and $(0, 1)$

(c) (i) $f'(x) = \dfrac{(x^2+5x+4)(2x-5)-(x^2-5x+4)(2x+5)}{(x^2+5x+4)^2}$ M1A1A1

$= \dfrac{10x^2-40}{(x^2+5x+4)^2}\ \left(= \dfrac{10(x-2)(x+2)}{(x^2+5x+4)^2}\right)$ A1

$f'(x) = 0 \Rightarrow x = \pm 2$ M1

so the points under consideration are $(-2, -9)$ and $\left(2, -\dfrac{1}{9}\right)$ A1A1

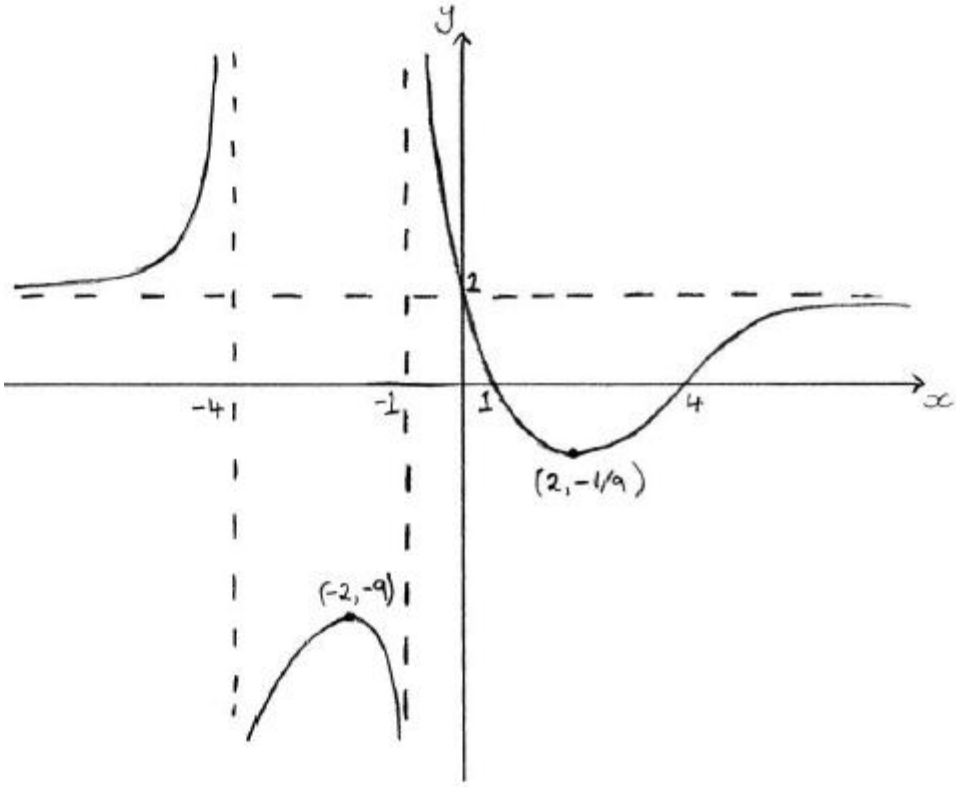
looking at the sign either side of the points (or attempt to find $f''(x)$) M1

e.g. if $x = -2^-$ then $(x - 2)(x + 2) > 0$ and if $x = -2^+$ then $(x - 2)(x + 2) < 0$, therefore $(-2, -9)$ is a maximum A1

(ii) e.g. if $x = 2^-$ then $(x - 2)(x + 2) < 0$ and if $x = 2^+$ then $(x - 2)(x + 2) > 0$, therefore $\left(2, -\dfrac{1}{9}\right)$ is a minimum A1

Note: Candidates may find the minimum first.

(d)



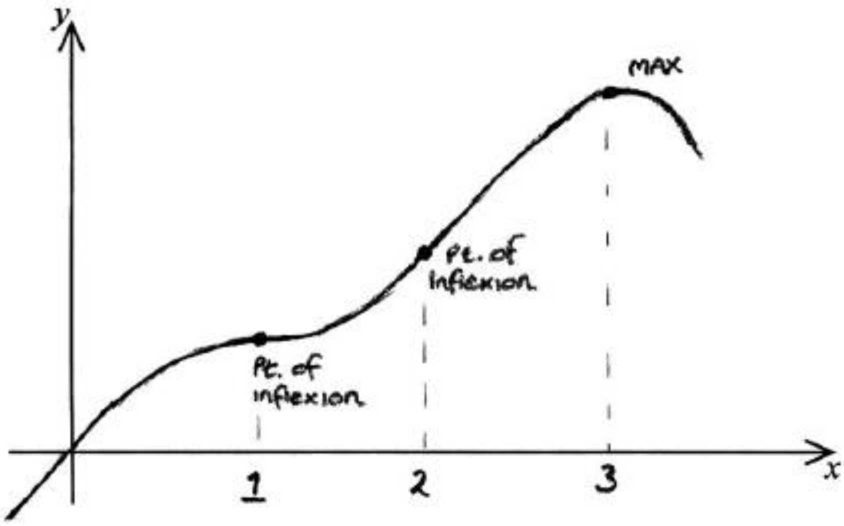
A3

Note: Award A1 for each branch consistent with and including the features found in previous parts.

(e) one A1

[20]

5.



A5

Note: Award A1 for origin
A1 for shape
A1 for maximum
A1 for each point of inflexion.

[5]

6. (a) $f'(x) = (1 + 2x)\,e^{2x}$ A1

$f'(x) = 0$ M1

$\Rightarrow (1 + 2x)e^{2x} = 0 \Rightarrow x = -\dfrac{1}{2}$ A1

$f''(x) = (2^2x + 2 \times 2^{2-1})e^{2x} = (4x + 4)e^{2x}$ A1

$f''\left(-\dfrac{1}{2}\right) = \dfrac{2}{e}$ A1

$\dfrac{2}{e} > 0 \Rightarrow$ at $x = -\dfrac{1}{2}$, $f(x)$ has a minimum. R1

$P\left(-\dfrac{1}{2}, -\dfrac{1}{2e}\right)$ A1

(b) $f''(x) = 0 \Rightarrow 4x + 4 = 0 \Rightarrow x = -1$ M1A1

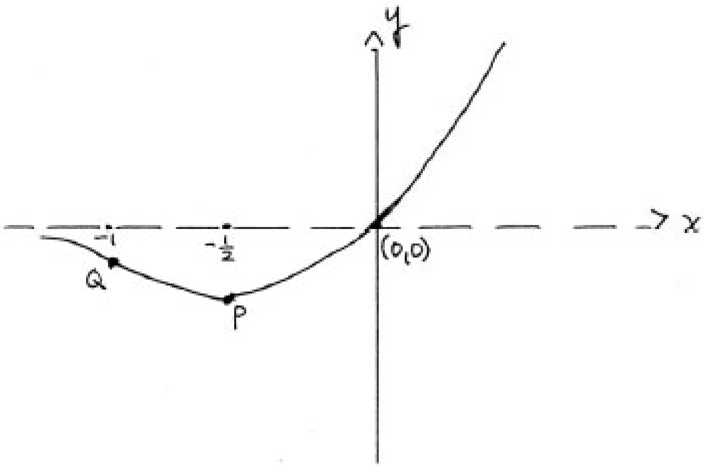
Using the 2nd derivative $f''\left(-\dfrac{1}{2}\right) = \dfrac{2}{e}$ and $f''(-2) = -\dfrac{4}{e^4}$, M1A1

the sign change indicates a point of inflexion. R1

(c) (i) $f(x)$ is concave up for $x > -1$. A1

(ii) $f(x)$ is concave down for $x < -1$. A1

(d)



A1A1A1A1

Note: Award A1 for P and Q, with Q above P,
A1 for asymptote at y = 0,
A1 for (0, 0),
A1 for shape.

(e) Show true for $n = 1$ (M1)

$f'(x) = e^{2x} + 2xe^{2x}$ A1

$= e^{2x}(1 + 2x) = (2x + 2^0)\,e^{2x}$

Assume true for $n = k$, ie $f^{(k)}\,x = (2^k x + k \times 2^{k-1})\,e^{2x}$, $k \geq 1$ M1A1

Consider $n = k + 1$, ie an attempt to find $\frac{d}{dx}\left(f^k(x)\right)$. M1

$f^{(k+1)}(x) = 2^k e^{2x} + 2e^{2x}(2^k x + k \times 2^{k-1})$ A1

$= (2^k + 2(2^k x + k \times 2^{k-1}))\,e^{2x}$

$= (2 \times 2^k x + 2^k + k \times 2 \times 2^{k-1})\,e^{2x}$

$= (2^{k+1} x + 2^k + k \times 2^k)\,e^{2x}$ A1

$= (2^{k+1} x + (k + 1)\,2^k)\,e^{2x}$ A1

$P(n)$ is true for $k \Rightarrow P(n)$ is true for $k + 1$, and since true

for $n = 1$, result proved by mathematical induction $\forall n \in \mathbb{Z}^+$ R1

Note: Only award R1 if a reasonable attempt is made to prove the $(k + 1)^{\text{th}}$ step.

[27]

7. (a) (i) $f_k'(x) = 3k^2x^2 - 2kx + 1$ A1

$f_k''(x) = 6k^2x - 2k$ A1

(ii) Setting $f''(x) = 0$ M1

$\Rightarrow 6k^2x - 2k = 0 \Rightarrow x = \frac{1}{3k}$ A1

$f\left(\frac{1}{3k}\right) = k^2\left(\frac{1}{3k}\right)^3 - k\left(\frac{1}{3k}\right)^2 + \left(\frac{1}{3k}\right)$ M1

$= \frac{7}{27k}$ A1

Hence, P_k is $\left(\frac{1}{3k}, \frac{7}{27k}\right)$

(b) Equation of the straight line is $y = \frac{7}{9}x$ A1

As this equation is independent of k, all P_k lie on this straight line R1

(c) Gradient of tangent at P_k:

$$f'(P_k) = f'\left(\frac{1}{3k}\right) = 3k^2\left(\frac{1}{3k}\right)^2 - 2k\left(\frac{1}{3k}\right) + 1 = \frac{2}{3}$$ M1A1

As the gradient is independent of k, the tangents are parallel. R1

$$\frac{7}{27k} = \frac{2}{3} \times \frac{1}{3k} + c \Rightarrow c = \frac{1}{27k}$$ (A1)

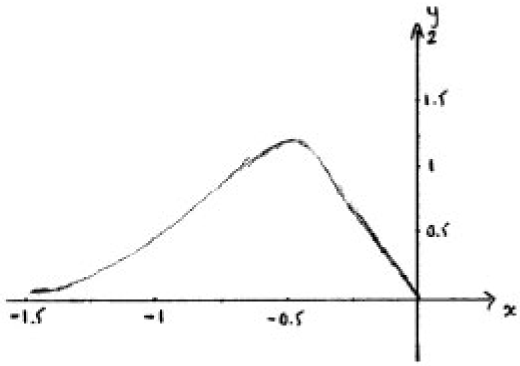
The equation is $y = \frac{2}{3}x + \frac{1}{27k}$ A1

[13]

8. **METHOD 1**

EITHER

Using the graph of $y = f'(x)$ (M1)

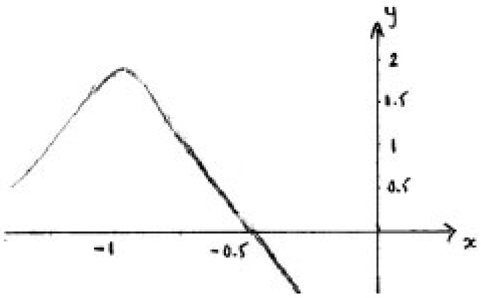


A1

The maximum of $f'(x)$ occurs at $x = -0.5$. A1

OR

Using the graph of $y = f''(x)$. (M1)



A1

The zero of $f''(x)$ occurs at $x = -0.5$. A1

THEN

Note: Do not award this A1 for stating $x = \pm\, 0.5$ as the final answer for x.

$f\,(-0.5) = 0.607$ $(= e^{-0.5})$ A2

Note: Do not award this A1 for also stating (0.5, 0.607) as a coordinate.

EITHER

Correctly labelled graph of $f'(x)$ for $x < 0$ denoting the maximum $f'(x)$ R1
(e.g. $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated) A1 N2

OR

Correctly labelled graph of $f''(x)$ for $x < 0$ denoting the maximum $f'(x)$ R1
(e.g. $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated) A1 N2

OR

$f'(0.5) \approx 1.21$. $f'(x) < 1.21$ just to the left of $x = -\frac{1}{2}$

and $f'(x) < 1.21$ just to the right of $x = -\frac{1}{2}$ R1

(e.g. $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated) A1 N2

OR

$f''(x) > 0$ just to the left of $x = -\frac{1}{2}$ and $f''(x) < 0$ just to the right

of $x = -\frac{1}{2}$ R1

(e.g. $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated) A1 N2

METHOD 2

$f'(x) = -4x\,e^{-2x^2}$ A1

$f''(x) = -4\,e^{-2x^2} + 16x^2\,e^{-2x^2}$ $\left(= \left(16x^2 - 4\right)e^{-2x^2}\right)$ A1

Attempting to solve $f''(x) = 0$ (M1)

$x = -\frac{1}{2}$ A1

Note: Do not award this A1 for stating $x = \pm\frac{1}{2}$ as the final answer for x.

$f\left(-\frac{1}{2}\right) = \frac{1}{\sqrt{e}}$ $(= 0.607)$ A1

Note: Do not award this A1 for also stating $\left(\frac{1}{2}, \frac{1}{\sqrt{e}}\right)$ as a coordinate.

EITHER

Correctly labelled graph of $f'(x)$ for $x < 0$ denoting the maximum $f'(x)$ R1

(e.g. $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated) A1 N2

OR

Correctly labelled graph of $f''(x)$ for $x < 0$ denoting the maximum $f'(x)$ R1

(e.g. $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated) A1 N2

OR

$f'(0.5) \approx 1.21$. $f'(x) < 1.21$ just to the left of $x = -\frac{1}{2}$

and $f'(x) < 1.21$ just to the right of $x = -\frac{1}{2}$ R1

(e.g. $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated) A1 N2

OR

$f''(x) > 0$ just to the left of $x = -\frac{1}{2}$ and $f''(x) < 0$ just to the right

of $x = -\frac{1}{2}$ R1

(e.g. $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated) A1 N2

[7]

9. $f'(x) = \dfrac{2(\ln(x-2))}{x-2}$ M1A1

$f''(x) = \dfrac{(x-2)\left(\dfrac{1}{x-2}\right) - 2\ln(x-2)\times 1}{(x-2)^2}$ M1A1

$= \dfrac{2-2\ln(x-2)}{(x-2)^2}$ A1

$f''(x) = 0$ for point of inflexion (M1)

$\Rightarrow\ 2 - 2\ln(x-2) = 0$

$\ln(x-2) = 1$ A1

$x - 2 = \mathrm{e}$

$x = \mathrm{e} + 2$ A1

$\Rightarrow\ f(x) = (\ln(\mathrm{e} + 2 - 2))^2 = (\ln \mathrm{e})^2 = 1$ A1

$(\Rightarrow$ coordinates are $(\mathrm{e} + 2, 1))$

[9]

10. (a) (i) $18(x-1) = 0\ \Rightarrow\ x = 1$ A1

(ii) vertical asymptote: $x = 0$ A1

horizontal asymptote: $y = 0$ A1

(iii) $18(2-x) = 0\ \Rightarrow\ x = 2$ M1A1

$f''(2) = \dfrac{36(2-3)}{2^3} = -\dfrac{9}{2} < 0$ hence it is a maximum point R1

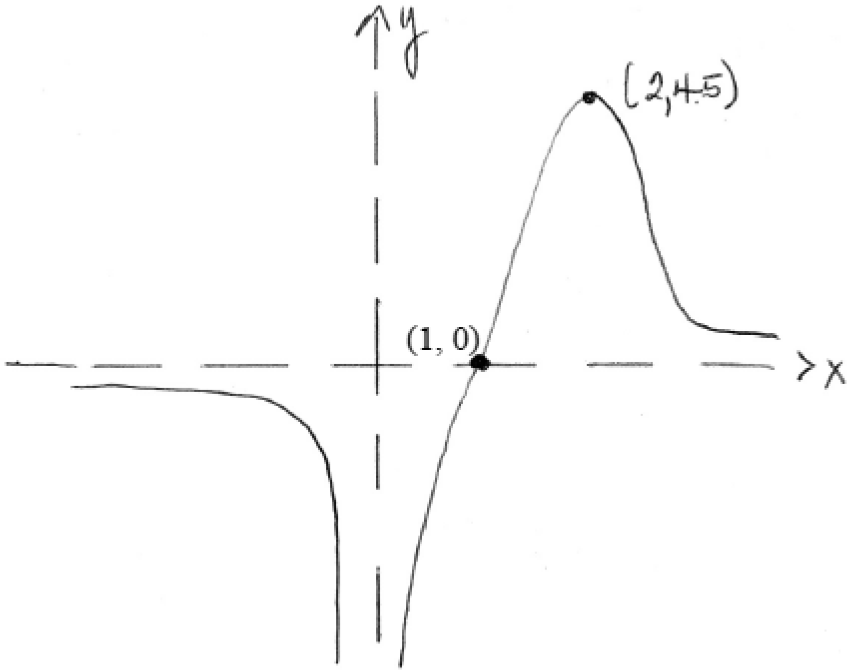
When $x = 2$, $f(x) = \dfrac{9}{2}$ A1

$\left(f(x) \text{ has a maximum at } \left(2, \dfrac{9}{2}\right)\right)$

(iv) $f(x)$ is concave up when $f''(x) > 0$ M1

$36(x-3) > 0\ \Rightarrow\ x > 3$ A1

(b)



A1A1A1A1A1

Note: Award A1 for shape, A1 for maximum, A1 for x-intercept, A1 for horizontal asymptote and A1 for vertical asymptote.

[14]

11. (a) (i) Attempting to use quotient rule $f'(x) = \dfrac{x\frac{1}{x} - \ln x \times 1}{x^2}$ (M1)

$f'(x) = \dfrac{1-\ln x}{x^2}$ A1

$f''(x) = \dfrac{x^2\left(-\frac{1}{x}\right)-(1-\ln x)2x}{x^4}$ (M1)

$f''(x) = \dfrac{2\ln x - 3}{x^3}$ A1

Stationary point where $f'(x) = 0$ M1

i.e. $\ln x = 1$, (so $x = \text{e}$) A1

$f''(\text{e}) < 0$ so maximum. R1AG N0

(ii) Exact coordinates $x = \text{e}$, $y = \dfrac{1}{\text{e}}$ A1A1 N2

(iii) Solving $f''(0) = 0$ M1

$\ln x = \frac{3}{2}$ (A1)

$x = e^{\frac{3}{2}}$ A1 N2

(b) Area $= \int_1^5 \frac{\ln x}{x} dx$ A1

EITHER

Finding the integral by substitution/inspection

$u = \ln x$, $du = \frac{1}{x} dx$ (M1)

$\int u du = \frac{u^2}{2} \quad \left(= \frac{(\ln x)^2}{2} \right)$ M1A1

Area $= \left[\frac{(\ln x)^2}{2} \right]_1^5 = \frac{1}{2}\left((\ln 5)^2 - (\ln 1)^2 \right)$ A1

Area $= \frac{1}{2}(\ln 5)^2$ A1 N2

OR

Finding the integral I by parts (M1)

$u = \ln x$, $dv = \frac{1}{x} \Rightarrow du = \frac{1}{x}$, $v = \ln x$

$I = uv - \int u dv = (\ln x)^2 - \int \ln x \frac{1}{x} dx = (\ln x)^2 - I$ M1

$\Rightarrow 2I = (\ln x)^2 \Rightarrow I = \frac{(\ln x)^2}{2}$ A1

Area $= \left[\frac{(\ln x)^2}{2} \right]_1^5 = \frac{1}{2}\left((\ln 5)^2 - (\ln 1)^2 \right)$ A1

Area $= \frac{1}{2}(\ln 5)^2$ A1 N2

[18]

12. (a) $\ln e^{2-2x} = \ln 2e^{-x}$ M1

$2 - 2x = \ln(2e^{-x})$ (A1)

$= \ln 2 - x$ (A1)

$x = 2 - \ln 2$ A1

$\left(x = \ln e^2 - \ln 2 = \ln \frac{e^2}{2} \right)$

(b) $\frac{dy}{dx} = -2e^{2-2x} + 2e^{-x}$ M1A1

$\frac{dy}{dx} = 0$ for a minimum point (M1)

$-2e^{2-2x} + 2e^{-x} = 0$

$\Rightarrow\ e^{2-2x} = e^{-x}$ (A1)

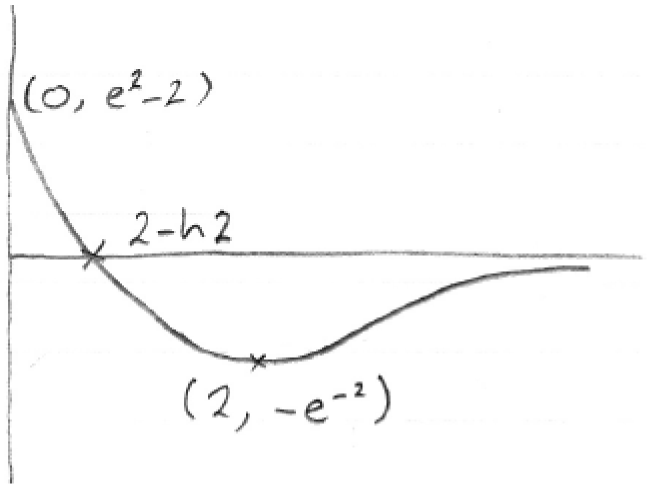
$\Rightarrow\ 2 - 2x = -x$ (A1)

$\Rightarrow\ x = 2$ A1

$\Rightarrow\ y = e^{-2} - 2e^{-2} = -e^{-2}$ A1

($\Rightarrow$ minimum point is $(2, -e^{-2})$)

(c)



A1A1A1

(d) 2 distinct roots provided $-e^{-2} < k < 0$ A1A1

[16]

13. (a) $f'(x) = \frac{2xe^x - x^2e^x}{e^{2x}} \left(= \frac{2x - x^2}{e^x}\right)$ M1A1

For a maximum $f'(x) = 0$ (M1)

$2x - x^2 = 0$

giving $x = 0$ or 2 A1A1

$f''(x) = \frac{(2-2x)e^x - e^x(2x - x^2)}{e^{2x}} \left(= \frac{x^2 - 4x + 2}{e^x}\right)$ M1A1

$f''(0) = 2 > 0 \Rightarrow$ minimum R1

$f''(2) = -\frac{2}{e^2} < 0 \Rightarrow$ maximum R1

Maximum value $= \frac{4}{e^2}$ A1

(b) For a point of inflexion,

$f''(x) = \frac{x^2 - 4x + 2}{e^x} = 0$ M1

giving $x = \frac{4 \pm \sqrt{16 - 8}}{2}$ (A1)

$= 2 \pm \sqrt{2}$ A1

(c) $\int_0^1 x^2 e^{-x} dx = \left[-x^2 e^{-x}\right]_0^1 + 2\int_0^1 x e^{-x} dx$ M1A1

$= -e^{-1} - 2\left[x e^{-x}\right]_0^1 + 2\int_0^1 e^{-x} dx$ A1M1A1

$= -e^{-1} - 2e^{-1} - 2\left[e^{-x}\right]_0^1$ A1A1

$= -3e^{-1} - 2e^{-1} + 2$ $(= 2 - 5e^{-1})$ A1

[21]