1. (a) Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^3$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$.

(2)

(b) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(3)

(c) Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(3)

(d) **Hence** solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(6)

(e) By considering the solutions of the equation $\cos 5\theta = 0$, show that

$\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$.

(8)
(Total 22 marks)

2. (a) Factorize $z^3 + 1$ into a linear and quadratic factor.

(2)

Let $\gamma = \frac{1+\mathrm{i}\sqrt{3}}{2}$.

(b) (i) Show that γ is one of the cube roots of -1.

(ii) Show that $\gamma^2 = \gamma - 1$.

(iii) Hence find the value of $(1 - \gamma)^6$.

(9)

The matrix $\boldsymbol{A}$ is defined by $\boldsymbol{A} = \begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix}$.

(c) Show that $\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I} = \boldsymbol{0}$, where $\boldsymbol{0}$ is the zero matrix.

(4)

(d) Deduce that

(i) $\boldsymbol{A}^3 = -\boldsymbol{I}$;

(ii) $\boldsymbol{A}^{-1} = \boldsymbol{I} - \boldsymbol{A}$.

(5)
(Total 20 marks)

3. Consider $\omega = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$.

(5)

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \frac{\pi}{2}$ on an Argand diagram.

(4)

(c) (i) Expand and simplify $F(z) = (z - 1)(z - \omega)(z - \omega^2)$ where z is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω.

(7)
(Total 16 marks)

4. (a) Solve the equation $z^3 = -2 + 2\text{i}$, giving your answers in modulus–argument form.

(6)

(b) **Hence** show that one of the solutions is $1 + \text{i}$ when written in Cartesian form.

(1)
(Total 7 marks)

5. The complex number z is defined as $z = \cos\theta + \text{i}\sin\theta$.

(a) State de Moivre's theorem.

(1)

(b) Show that $z^n - \frac{1}{z^n} = 2\text{i}\sin(n\theta)$.

(3)

(c) Use the binomial theorem to expand $\left(z - \frac{1}{z}\right)^5$ giving your answer in simplified form.

(3)

(d) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(4)

(e) Check that your result in part (d) is true for $\theta = \frac{\pi}{4}$.

(4)

(f) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta \, \text{d}\theta$.

(4)

(g) Hence, with reference to graphs of circular functions, find $\int_0^{\frac{\pi}{2}} \cos^5 \theta \, d\theta$, explaining your reasoning.

(3)
(Total 22 marks)

6. Find the values of n such that $(1 + \sqrt{3}\,i)^n$ is a real number.

(Total 5 marks)

7. (a) Let $z = x + iy$ be any non-zero complex number.

(i) Express $\frac{1}{z}$ in the form $u + iv$.

(ii) If $z + \frac{1}{z} = k, k \in \mathbb{R}$, show that either $y = 0$ or $x^2 + y^2 = 1$.

(iii) Show that if $x^2 + y^2 = 1$ then $|k| \le 2$.

(8)

(b) Let $w = \cos \theta + i \sin \theta$.

(i) Show that $w^n + w^{-n} = 2\cos n\theta, n \in \mathbb{Z}$.

(ii) Solve the equation $3w^2 - w + 2 - w^{-1} + 3w^{-2} = 0$, giving the roots in the form $x + iy$.

(14)
(Total 22 marks)

8. Express $\frac{1}{\left(1 - i\sqrt{3}\right)^3}$ in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$.

(Total 5 marks)

9. Let $w = \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$.

(a) Show that w is a root of the equation $z^5 - 1 = 0$.

(3)

(b) Show that $(w - 1)(w^4 + w^3 + w^2 + w + 1) = w^5 - 1$ and deduce that $w^4 + w^3 + w^2 + w + 1 = 0$.

(3)

(c) **Hence** show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

(6)
(Total 12 marks)

10. $z_1 = \left(1+\mathrm{i}\sqrt{3}\right)^m$ and $z_2 = (1-\mathrm{i})^n$.

(a) Find the modulus and argument of z_1 and z_2 in terms of m and n, respectively.

(6)

(b) **Hence**, find the smallest positive integers m and n such that $z_1 = z_2$.

(8)
(Total 14 marks)

11. (a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \mathrm{i}$.

(6)

(b) Draw these roots on an Argand diagram.

(2)

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\frac{z_2}{z_1}$ in the form $a + \mathrm{i}b$.

(4)
(Total 12 marks)

12. Find the three cube roots of the complex number 8i. Give your answers in the form $x + \mathrm{i}y$.

(Total 8 marks)

13. The roots of the equation $z^2 + 2z + 4 = 0$ are denoted by α and β?

(a) Find α and β in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

(6)

(b) Given that α lies in the second quadrant of the Argand diagram, mark α and β on an Argand diagram.

(2)

(c) Use the principle of mathematical induction to prove De Moivre's theorem, which states that $\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n$ for $n \in \mathbb{Z}^+$.

(8)

(d) Using De Moivre's theorem find $\frac{\alpha^3}{\beta^2}$ in the form $a + \mathrm{i}b$.

(4)

(e) Using De Moivre's theorem or otherwise, show that $\alpha^3 = \beta^3$.

(3)

(f) Find the exact value of $\alpha\beta^* + \beta\alpha^*$ where α^* is the conjugate of α and β^* is the conjugate of β.

(5)

(g) Find the set of values of n for which α^n is real.

(3)
(Total 31 marks)