OPTIMIZATION - SOLUTIONS

1. (a) $\left(u = x^2 - 2x - 1.5; \frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 2\right)$

$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x(2x - 2)$ (M1)

$= 2(x - 1)\,\mathrm{e}^{x^2 - 2x - 1.5}$ A1

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-1) \times 2(x-1)\mathrm{e}^{x^2-2x-1.5} - 1 \times \mathrm{e}^{x^2-2x-1.5}}{(x-1)^2}$ M1A1

$= \frac{2x^2 - 4x + 1}{(x+1)^2}\mathrm{e}^{x^2-2x-1.5}$ (A1)

minimum occurs when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ (M1)

$x = 1 \pm \sqrt{\frac{1}{2}}$ $\left(\text{accept } x = \frac{4 \pm \sqrt{8}}{4}\right)$ A1

$a = 1 + \sqrt{\frac{1}{2}}$ $\left(\text{accept } a = \frac{4 + \sqrt{8}}{4}\right)$ R1

[8]

2. (a) $\mathrm{O\hat{A}B} = \pi - \theta$ (allied) A1

recognizing OAB as an isosceles triangle M1

so $\mathrm{A\hat{B}O} = \pi - \theta$ A1

$\mathrm{B\hat{O}C} = \pi - \theta$ (alternate) AG

Note: This can be done in many ways, including a clear diagram.

(b) area of trapezium is $T = \text{area}_{\Delta\mathrm{BOC}} + \text{area}_{\Delta\mathrm{AOB}}$ (M1)

$= \frac{1}{2}r^2 \sin(\pi - \theta) + \frac{1}{2}r^2 \sin(2\theta - \pi)$ M1A1

$= \frac{1}{2}r^2 \sin\theta - \frac{1}{2}r^2 \sin 2\theta$ AG

(c) (i) $\frac{\mathrm{d}T}{\mathrm{d}\theta} = \frac{1}{2}r^2 \cos\theta - r^2 \cos 2\theta$ M1A1

for maximum area $\frac{1}{2}r^2 \cos\theta - r^2 \cos 2\theta = 0$ M1

$\cos\theta = 2\cos 2\theta$ AG

(ii) $\theta_{\max} = 2.205...$ (A1)

$\frac{1}{2}\sin\theta_{\max} - \frac{1}{2}\sin 2\theta_{\max} = 0.880$ A1

[11]

3. (a) solving to obtain one root: 1, –2 or –5 A1
obtain other roots A1

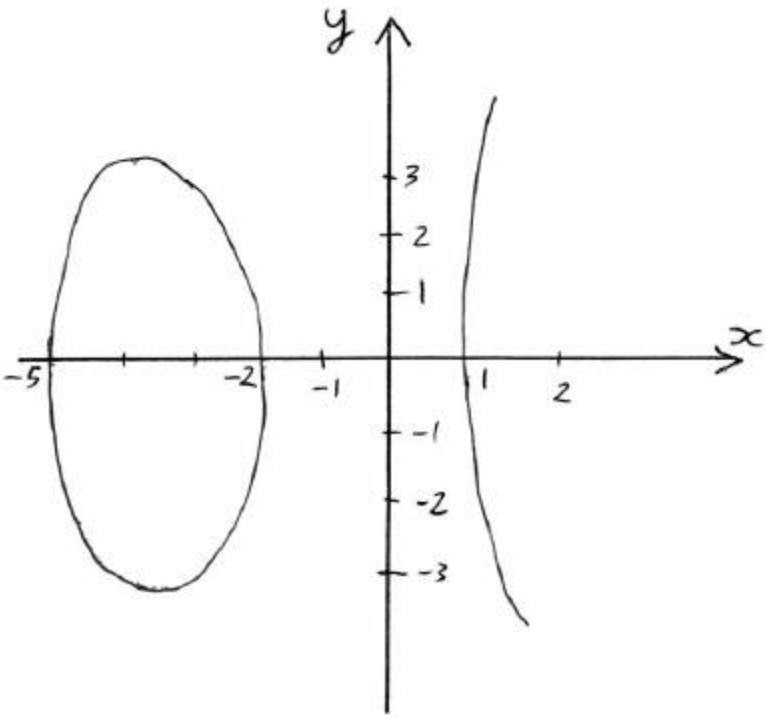
(b) $D = x \in [-5, -2] \cup [1, \infty)$ (or equivalent) M1A1

Note: M1 is for 1 finite and 1 infinite interval.

(c) coordinates of local maximum –3.73 ($-2-\sqrt{3}$), 3.22($\sqrt{6\sqrt{3}}$) A1A1

(d) use GDC to obtain one root: 1.41, –3.18 or –4.23 A1
obtain other roots A1

(e)



A1A1A1

Note: Award A1 for shape, A1 for max and for min clearly in correct places, A1 for all intercepts.

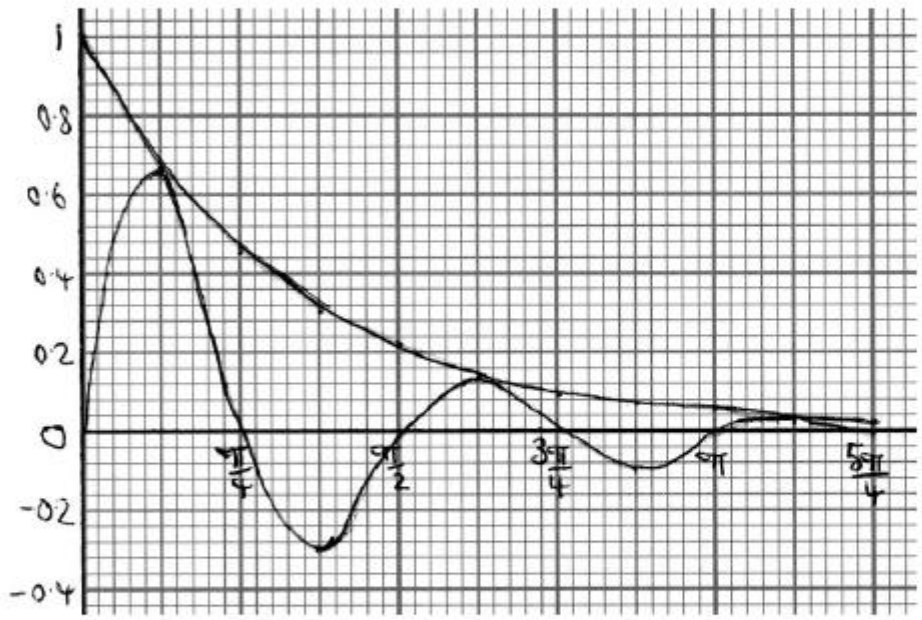
Award A1A0A0 if only the complete top half is shown.

(f) required area is twice that of $y = f(x)$ between –5 and –2 M1A1
answer 14.9 A1 N3

Note: Award M1A0A0 for $\int_{-5}^{-2} f(x)\mathrm{d}x = 7.47...$ or N1 for 7.47.

[14]

4. (a)



A3

Note: Award A1 for each correct **shape**,
A1 for correct relative position.

(b) $e^{-x} \sin(4x) = 0$ (M1)
$\sin(4x) = 0$ A1
$4x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$ A1
$x = 0, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}, \frac{5\pi}{4}$ AG

(c) $e^{-x} = e^{-x} \sin(4x)$ or reference to graph
$\sin 4x = 1$ M1
$4x = \frac{\pi}{4}, \frac{5\pi}{2}, \frac{9\pi}{2}$ A1
$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$ A1 N3

(d) (i) $y = e^{-x}\sin 4x$

$\frac{dy}{dx} = -e^{-x}\sin 4x + 4e^{-x}\cos 4x$ M1A1

$y = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$ A1

verifying equality of gradients at one point R1

verifying at the other two R1

(ii) since $\frac{dy}{dx} \neq 0$ at these points they cannot be local maxima R1

(e) (i) maximum when $y' = 4e^{-x}\cos 4x - e^{-x}\sin 4x = 0$ M1

$x = \frac{\arctan(4)}{4}, \frac{\arctan(4)+\pi}{4}, \frac{\arctan(4)+2\pi}{4}, \ldots$

maxima occur at

$x = \frac{\arctan(4)}{4}, \frac{\arctan(4)+2\pi}{4}, \frac{\arctan(4)+4\pi}{4}$ A1

so $y_1 = e^{-\frac{1}{4}(\arctan(4))}\sin(\arctan(4))$ (= 0.696) A1

$y_2 = e^{-\frac{1}{4}(\arctan(4)+2\pi)}\sin(\arctan(4) + 2\pi)$ A1

$\left(= e^{-\frac{1}{4}(\arctan(4)+2\pi)}\sin(\arctan(4)) = 0.145\right)$

$y_3 = e^{-\frac{1}{4}(\arctan(4)+4\pi)}\sin(\arctan(4) + 4\pi)$ A1

$\left(= e^{-\frac{1}{4}(\arctan(4)+4\pi)}\sin(\arctan(4)) = 0.0301\right)$ N3

(ii) for finding and comparing $\frac{y_3}{y_2}$ and $\frac{y_2}{y_1}$ M1

$r = e^{-\frac{\pi}{2}}$ A1

Note: Exact values must be used to gain the M1 and the A1.

[22]

5. (a) (i) $x - a\sqrt{x} = 0$ M1

$\sqrt{x}\sqrt{x} - a = 0$ (A1)

$x = 0, x = a^2$ A1 N2

(ii) $f'(x) = 1 - \frac{a}{2\sqrt{x}}$ A1

f is decreasing when $f' < 0$ (M1)

$1-\frac{a}{2\sqrt{x}} < 0 \Rightarrow \frac{2\sqrt{x}-a}{2\sqrt{x}} < 0 \Rightarrow x < \frac{a^4}{4}$ A1

(iii) f is increasing when $f' > 0$

$1-\frac{a}{2\sqrt{x}} > 0 \Rightarrow \frac{2\sqrt{x}-a}{2\sqrt{x}} > 0 \Rightarrow x > \frac{a^4}{4}$ A1

Note: Award the M1 mark for either (ii) or (iii).

(iv) minimum occurs at $x = \frac{a^4}{4}$

minimum value is $y = -\frac{a^2}{4}$ (M1)A1

hence $y \geq -\frac{a^2}{4}$ A1

(b) concave up for all values of x R1

[11]

6. (a) $f'(x)=1-\frac{2}{x^{\frac{1}{3}}}$ A1

$\Rightarrow 1-\frac{2}{x^{\frac{1}{3}}}=0 \Rightarrow x^{\frac{1}{3}}=2 \Rightarrow x=8$ A1

(b) $f''(x)=\frac{2}{3x^{\frac{4}{3}}}$ A1

$f''(8) > 0 \Rightarrow$ at $x = 8$, $f(x)$ has a minimum. M1A1

[5]

7. (a) AQ = $\sqrt{x^2+4}$ (km) (A1)

QY = $(2 - x)$ (km) (A1)

$T = 5\sqrt{5}\,\text{AQ} + 5\text{QY}$ (M1)

$= 5\sqrt{5}\sqrt{(x^2+4)} + 5(2-x)$ (mins) A1

(b) Attempting to use the chain rule on $5\sqrt{5}\sqrt{(x^2+4)}$ (M1)

$\frac{\text{d}}{\text{d}x}\left(5\sqrt{5}\sqrt{(x^2+4)}\right) = 5\sqrt{5} \times \frac{1}{2}\left(x^2+4\right)^{-\frac{1}{2}} \times 2x$ A1

$\left(= \frac{5\sqrt{5}x}{\sqrt{x^2+4}}\right)$

$\frac{\text{d}}{\text{d}x}\left(5(2-x)\right) = -5$ A1

$\frac{\text{d}T}{\text{d}x} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5$ AG N0

(c) (i) $\sqrt{5}\,x = \sqrt{x^2+4}$ or equivalent A1

Squaring both sides and rearranging to obtain $5x^2 = x^2 + 4$ M1

$x = 1$ A1 N1

Note: Do not award the final A1 for stating a negative solution in final answer.

(ii) $T = 5\sqrt{5}\sqrt{1+4} + 5(2-1)$ M1

= 30 (mins) A1 N1

Note: Allow FT on incorrect x value.

(iii) **METHOD 1**

Attempting to use the quotient rule M1

$u = x, v = \sqrt{x^2+4}, \frac{du}{dx} = 1$ and $\frac{dv}{dx} = x(x^2+4)^{-\frac{1}{2}}$ (A1)

$$\frac{d^2T}{dx^2} = 5\sqrt{5}\left[\frac{\sqrt{x^2+4} - \frac{1}{2}(x^2+4)^{-\frac{1}{2}} \times 2x^2}{(x^2+4)}\right]$$ A1

Attempt to simplify (M1)

$$= \frac{5\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}\left[x^2+4-x^2\right]$$ or equivalent A1

$$= \frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}$$ AG

When $x = 1$, $\frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}} > 0$ and hence $T = 30$ is a minimum R1 N0

Note: Allow FT on incorrect x value, $0 \le x \le 2$.

METHOD 2

Attempting to use the product rule M1

$u = x, v = \sqrt{x^2+4}, \frac{du}{dx} = 1$ and $\frac{dv}{dx} = x(x^2+4)^{-\frac{1}{2}}$ (A1)

$$\frac{d^2T}{dx^2} = 5\sqrt{5}\,(x^2+4)^{-\frac{1}{2}} - \frac{5\sqrt{5}x}{2}(x^2+4)^{-\frac{3}{2}} \times 2x$$ A1

$$\left(= \frac{5\sqrt{5}}{(x^2+4)^{\frac{1}{2}}} - \frac{5\sqrt{5}\,x^2}{(x^2+4)^{\frac{3}{2}}} \right)$$

Attempt to simplify (M1)

$$= \frac{5\sqrt{5}(x^2+4) - 5\sqrt{5}\,x^2}{(x^2+4)^{\frac{3}{2}}} \quad \left(= \frac{5\sqrt{5}\,(x^2+4-x^2)}{(x^2+4)^{\frac{3}{2}}} \right)$$ A1

$$= \frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}$$ AG

When $x = 1$, $\frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}} > 0$ and hence $T = 30$ is a minimum R1 N0

Note: Allow FT on incorrect x value, $0 \le x \le 2$.

[18]

8. (a) Area of hexagon = $6 \times \frac{1}{2} \times x \times x \times \sin 60°$ M1

$= \frac{3\sqrt{3}x^2}{2}$ AG

(b) Let the height of the box be h

Volume = $\frac{3\sqrt{3}hx^2}{2} = 90$ M1

Hence $h = \frac{60}{\sqrt{3}x^2}$ A1

Surface area, $A = 3\sqrt{3}\,x^2 + 6hx$ M1

$= 3\sqrt{3}x^2 + \frac{360}{\sqrt{3}}x^{-1}$ A1

$\frac{\mathrm{d}A}{\mathrm{d}x} = 6\sqrt{3}x - \frac{360}{\sqrt{3}}x^{-2}$ A1

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} = 0\right)$

$6\sqrt{3}x^3 = \frac{360}{\sqrt{3}}$ M1

$x^3 = 20$

$x = \sqrt[3]{20}$ AG

$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 6\sqrt{3} + \frac{720x^{-3}}{\sqrt{3}}$

which is positive when $x = \sqrt[3]{20}$, and hence gives a minimum value. R1

[8]

9. (a) For $x\sqrt{9-x^2}$, $-3 \le x \le 3$ and for $2\arcsin\left(\frac{x}{3}\right)$, $-3 \le x \le 3$ A1

$\Rightarrow D$ is $-3 \le x \le 3$ A1

(b) $V = \pi\int_0^{2.8}\left(x\sqrt{9-x^2} + 2\arcsin\frac{x}{3}\right)^2\mathrm{d}x$ M1A1

$= 181$ A1

(c) $\dfrac{dy}{dx} = (9-x^2)^{\frac{1}{2}} - \dfrac{x^2}{(9-x^2)^{\frac{1}{2}}} + \dfrac{\frac{2}{3}}{\sqrt{1-\frac{x^2}{9}}}$ M1A1

$= (9-x^2)^{\frac{1}{2}} - \dfrac{x^2}{(9-x^2)^{\frac{1}{2}}} + \dfrac{2}{(9-x^2)^{\frac{1}{2}}}$ A1

$= \dfrac{9-x^2-x^2+2}{(9-x^2)^{\frac{1}{2}}}$ A1

$= \dfrac{11-2x^2}{\sqrt{9-x^2}}$ A1

(d) $\int_{-p}^{p} \dfrac{11-2x^2}{\sqrt{9-x^2}}dx = \left[x\sqrt{9-x^2} + 2\arcsin\dfrac{x}{3}\right]_{-p}^{p}$ M1

$= p\sqrt{9-p^2} + 2\arcsin\dfrac{p}{3} + p\sqrt{9-p^2} + 2\arcsin\dfrac{p}{3}$ A1

$= 2p\sqrt{9-p^2} + 4\arcsin\left(\dfrac{p}{3}\right)$ AG

(e) $11 - 2p^2 = 0$ M1

$p = 2.35$ $\left(\sqrt{\dfrac{11}{2}}\right)$ A1

Note: Award A0 for $p = \pm 2.35$.

(f) (i) $f''(x) = \dfrac{(9-x^2)^{\frac{1}{2}}(-4x) + x(11-2x^2)(9-x^2)^{-\frac{1}{2}}}{9-x^2}$ M1A1

$= \dfrac{-4x(9-x^2) + x(11-2x^2)}{(9-x^2)^{\frac{3}{2}}}$ A1

$= \dfrac{-36x + 4x^3 + 11x - 2x^3}{(9-x^2)^{\frac{3}{2}}}$ A1

$= \dfrac{x(2x^2-25)}{(9-x^2)^{\frac{3}{2}}}$ AG

(ii) **EITHER**

When $0 < x < 3$, $f''(x) < 0$. When $-3 < x < 0$, $f''(x) > 0$. A1

OR

$f''(0) = 0$ A1

THEN

Hence $f''(x)$ changes sign through $x = 0$, giving a point of inflexion. R1

EITHER

$x = \pm\sqrt{\dfrac{25}{2}}$ is outside the domain of f. R1

OR

$x = \pm\sqrt{\dfrac{25}{2}}$ is not a root of $f''(x) = 0$. R1

[21]

10. (a) $\text{OP} = \sqrt{a^2 + (a^2 - 5)^2}$ M1

$= \sqrt{a^2 + a^4 - 10a^2 + 25}$ A1

$= \sqrt{a^4 - 9a^2 + 25}$ AG

(b) **EITHER**

Let $s = \sqrt{a^4 - 9a^2 + 25}$

$\Rightarrow s^2 = a^4 - 9a^2 + 25$

$\dfrac{\mathrm{d}s^2}{\mathrm{d}a} = 4a^3 - 18a = 0$ M1A1

$\dfrac{\mathrm{d}s^2}{\mathrm{d}a} = 0$ for minimum (M1)

$\Rightarrow 2a(2a^2 - 9) = 0$

$\Rightarrow a^2 = \dfrac{9}{2}$

$\Rightarrow a = \pm\dfrac{3}{\sqrt{2}} \left(= \pm\dfrac{3\sqrt{2}}{2} \right)$ A1A1

OR

$s = (a^4 - 9a^2 + 25)^{\frac{1}{2}}$

$\frac{ds}{da} = \frac{1}{2}(a^4 - 9a^2 + 25)^{-\frac{1}{2}}(4a^3 - 18a)$ M1A1

$\frac{ds}{da} = 0$ for a minimum (M1)

$4a^3 - 18a = 0$

$\Rightarrow 2a(2a^2 - 9) = 0$

$\Rightarrow a^2 = \frac{9}{2}$

$\Rightarrow a = \pm\frac{3}{\sqrt{2}} \left(= \pm\frac{3\sqrt{2}}{2}\right)$ A1A1

[7]