OPTIMIZATION

1. The function f is defined by $f(x) = e^{x^2-2x-1.5}$.

(a) Find $f'(x)$.

(2)

(b) You are given that $y = \dfrac{f(x)}{x-1}$ has a local minimum at $x = a$, $a > 1$. Find the value of a.

(6)
(Total 8 marks)

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A trapezium OABC is formed such that AB is parallel to OC, and the angle $A\hat{O}C$ is θ, $\frac{\pi}{2} \le \theta < \pi$.

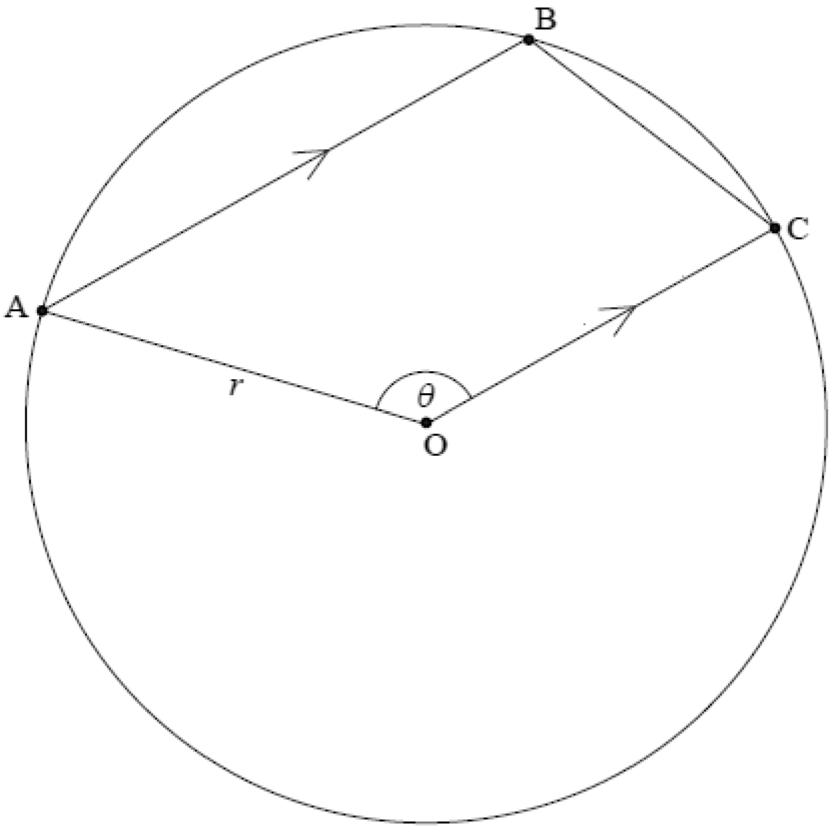


diagram not to scale

(a) Show that angle $\hat{BOC}$ is $\pi - \theta$.

(3)

(b) Show that the area, T, of the trapezium can be expressed as

$$T = \frac{1}{2}r^2 \sin\theta - \frac{1}{2}r^2 \sin 2\theta .$$

(3)

(c) (i) Show that when the area is maximum, the value of θ satisfies

$$\cos\theta = 2\cos 2\theta.$$

(ii) **Hence** determine the maximum area of the trapezium when $r = 1$.
(Note: It is not required to prove that it is a maximum.)

(5)
(Total 11 marks)

3. The function f is defined by

$$f(x) = (x^3 + 6x^2 + 3x - 10)^{\frac{1}{2}}, \text{ for } x \in D,$$

where $D \subseteq \mathbb{R}$ is the greatest possible domain of f.

(a) Find the roots of $f(x) = 0$.

(2)

(b) Hence specify the set D.

(2)

(c) Find the coordinates of the local maximum on the graph $y = f(x)$.

(2)

(d) Solve the equation $f(x) = 3$.

(2)

(e) Sketch the graph of $|y| = f(x)$, for $x \in D$.

(3)

(f) Find the area of the region completely enclosed by the graph of $|y| = f(x)$.

(3)
(Total 14 marks)

4. Consider the graphs $y = e^{-x}$ and $y = e^{-x} \sin 4x$, for $0 \le x \le \frac{5\pi}{4}$.

(a) On the same set of axes draw, on graph paper, the graphs, for $0 \le x \le \frac{5\pi}{4}$.

Use a scale of 1 cm to $\frac{\pi}{8}$ on your x-axis and 5 cm to 1 unit on your y-axis.

(3)

(b) Show that the x-intercepts of the graph $y = e^{-x} \sin 4x$ are $\frac{n\pi}{4}$, $n = 0, 1, 2, 3, 4, 5$.

(3)

(c) Find the x-coordinates of the points at which the graph of $y = e^{-x} \sin 4x$ meets the graph of $y = e^{-x}$. Give your answers in terms of π.

(3)

(d) (i) Show that when the graph of $y = e^{-x} \sin 4x$ meets the graph of $y = e^{-x}$, their gradients are equal.

(ii) Hence explain why these three meeting points are not local maxima of the graph $y = e^{-x} \sin 4x$.

(6)

(e) (i) Determine the y-coordinates, y_1, y_2 and y_3, where $y_1 > y_2 > y_3$, of the local maxima of $y = e^{-x} \sin 4x$ for $0 \le x \le \frac{5\pi}{4}$. You do not need to show that they are maximum values, but the values should be simplified.

(ii) Show that y_1, y_2 and y_3 form a geometric sequence and determine the common ratio r.

(7)
(Total 22 marks)

5. Consider the function f, defined by $f(x) = x - a\sqrt{x}$, where $x \ge 0$, $a \in \mathbb{R}^+$.

(a) Find in terms of a

(i) the zeros of f;

(ii) the values of x for which f is decreasing;

(iii) the values of x for which f is increasing;

(iv) the range of f.

(10)

(b) State the concavity of the graph of f.

(1)
(Total 11 marks)

6. If $f(x) = x - 3x^{\frac{2}{3}}$, $x > 0$,

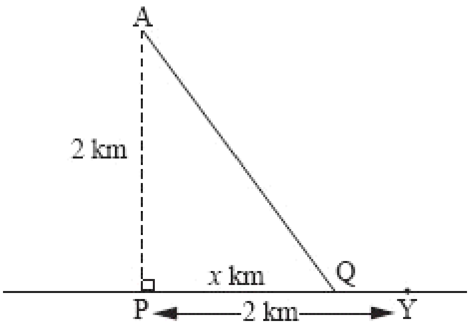
(a) find the x-coordinate of the point P where $f'(x) = 0$;

(2)

(b) determine whether P is a maximum or minimum point.

(3)
(Total 5 marks)

7. André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that AP = 2 km and PY = 2 km. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.



When André swims he covers 1 km in $5\sqrt{5}$ minutes. When he runs he covers 1 km in 5 minutes.

(a) If PQ = x km, $0 \le x \le 2$, find an expression for the time T minutes taken by André to reach point Y.

(4)

(b) Show that $\dfrac{\mathrm{d}T}{\mathrm{d}x} = \dfrac{5\sqrt{5}\,x}{\sqrt{x^2+4}} - 5.$

(3)

(c) (i) Solve $\dfrac{\mathrm{d}T}{\mathrm{d}x} = 0.$

(ii) Use the value of x found in **part (c) (i)** to determine the time, T minutes, taken for André to reach point Y.

(iii) Show that $\dfrac{\mathrm{d}^2T}{\mathrm{d}x^2} = \dfrac{20\sqrt{5}}{\left(x^2+4\right)^{\frac{3}{2}}}$ and **hence** show that the time found in **part (c) (ii)** is a minimum.

(11)
(Total 18 marks)

8. A packaging company makes boxes for chocolates. An example of a box is shown below. This box is closed and the top and bottom of the box are identical regular hexagons of side x cm.

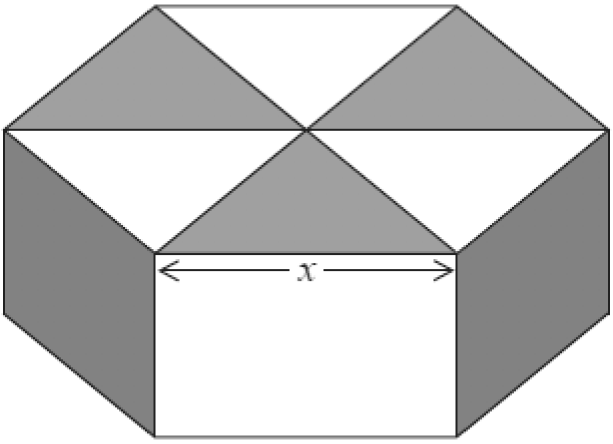


diagram not to scale

(a) Show that the area of each hexagon is $\dfrac{3\sqrt{3}x^2}{2}$ cm^2.

(1)

(b) Given that the volume of the box is 90 cm^3, show that when $x = \sqrt[3]{20}$ the total surface area of the box is a minimum, justifying that this value gives a minimum.

(7)
(Total 8 marks)

9. The function f is defined by $f(x) = x\sqrt{9-x^2} + 2\arcsin\left(\dfrac{x}{3}\right)$.

(a) Write down the largest possible domain, for each of the two terms of the function, f, and hence state the largest possible domain, D, for f.

(2)

(b) Find the volume generated when the region bounded by the curve $y = f(x)$, the x-axis, the y-axis and the line $x = 2.8$ is rotated through 2π radians about the x-axis.

(3)

(c) Find $f'(x)$ in simplified form.

(5)

(d) **Hence** show that $\int_{-p}^{p} \frac{11-2x^2}{\sqrt{9-x^2}}\mathrm{d}x = 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$, where $p \in D$.

(2)

(e) Find the value of p that maximises the value of the integral in (d).

(2)

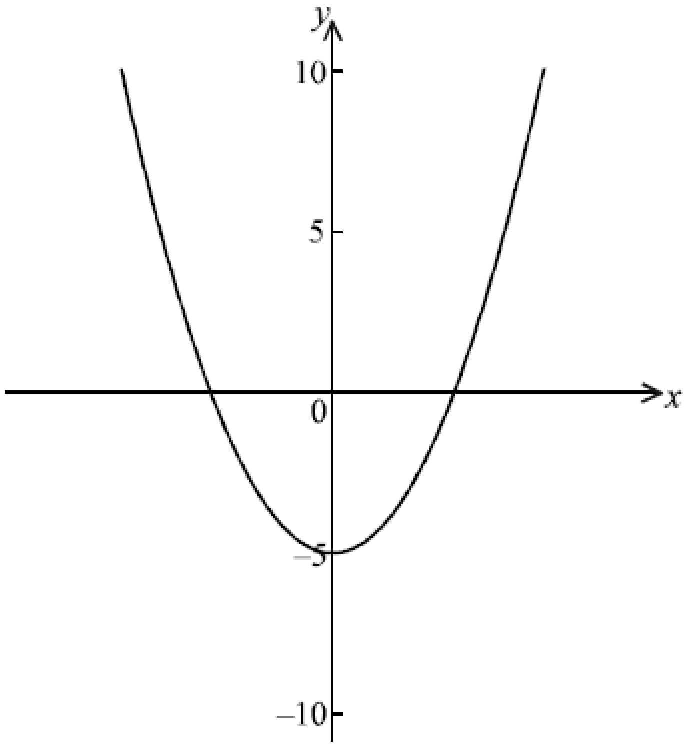
(f) (i) Show that $f''(x) = \frac{x(2x^2-25)}{(9-x^2)^{\frac{3}{2}}}$.

(ii) Hence justify that $f(x)$ has a point of inflexion at $x = 0$, but not at $x = \pm\sqrt{\frac{25}{2}}$.

(7)

(Total 21 marks)

10. The curve $y = x^2 - 5$ is shown below.



A point P on the curve has *x*-coordinate equal to *a*.

(a) Show that the distance OP is $\sqrt{a^4 - 9a^2 + 25}$.

(2)

(b) Find the values of *a* for which the curve is closest to the origin.

(5)
(Total 7 marks)