MATH HL2 EXAM PREP – CORE TOPICS – ALGEBRA (SOLUTIONS)

1. **METHOD 1**

$z = (2 - \text{i})(z + 2)$ M1

$\quad = 2z + 4 - \text{i}z - 2\text{i}$

$z(1 - \text{i}) = -4 + 2\text{i}$

$z = \dfrac{-4+2\text{i}}{1-\text{i}}$ A1

$z = \dfrac{-4+2\text{i}}{1-\text{i}} \times \dfrac{1+\text{i}}{1+\text{i}}$ M1

$= -3 - \text{i}$ A1

METHOD 2

let $z = a + \text{i}b$

$\dfrac{a+\text{i}b}{a+\text{i}b+2} = 2 - \text{i}$ M1

$a + \text{i}b = (2 - \text{i})((a + 2) + \text{i}b)$

$a + \text{i}b = 2(a + 2) + 2b\text{i} - \text{i}(a + 2) + b$

$a + \text{i}b = 2a + b + 4 + (2b - a - 2)\text{i}$

attempt to equate real and imaginary parts M1

$a = 2a + b + 4 (\Rightarrow a + b + 4 = 0)$

and $b = 2b - a - 2 (\Rightarrow -a + b - 2 = 0)$ A1

Note: Award A1 for two correct equations.

$b = -1; a = -3$

$z = -3 - \text{i}$ A1

[4]

2. (a) $u_1 = 27$

$\dfrac{81}{2} = \dfrac{27}{1-r}$ M1

$r = \dfrac{1}{3}$ A1

(b) $v_2 = 9$
$v_4 = 1$
$2d = -8 \Rightarrow d = -4$ (A1)
$v_1 = 13$ (A1)

$\frac{N}{2}(2 \times 13 - 4(N-1)) > 0$ (accept equality) M1

$\frac{N}{2}(30 - 4N) > 0$

$N(15 - 2N) > 0$

$N < 7.5$ (M1)

$N = 7$ A1

Note: $13 + 9 + 5 + 1 - 3 - 7 - 11 > 0 \Rightarrow N = 7$ or equivalent receives full marks.

[7]

3. prove that $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$

for $n = 1$

LHS = 1, RHS = $4 - \frac{1+2}{2^0} = 4 - 3 = 1$

so true for $n = 1$ R1

assume true for $n = k$ M1

so $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + k\left(\frac{1}{2}\right)^{k-1} = 4 - \frac{k+2}{2^{k-1}}$

now for $n = k + 1$

LHS: $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + k\left(\frac{1}{2}\right)^{k-1} + (k+1)\left(\frac{1}{2}\right)^k$ A1

$= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k$ M1A1

$= 4 - \frac{2(k+2)}{2^k} + \frac{k+1}{2^k}$ (or equivalent) A1

$= 4 - \frac{(k+1)+2}{2^{(k+1)-1}}$ (accept $4 - \frac{k+3}{2^k}$) A1

Therefore if it is true for $n = k$ it is true for $n = k + 1$. It has been shown to be true for $n = 1$ so it is true for all n ($\in \mathbb{Z}^+$). R1

Note: To obtain the final *R* mark, a reasonable attempt at induction must have been made.

[8]

4. P (six in first throw) = $\frac{1}{6}$ (A1)

P (six in third throw) = $\frac{25}{36} \times \frac{1}{6}$ (M1)(A1)

P (six in fifth throw) = $\left(\frac{25}{36}\right)^2 \times \frac{1}{6}$

P(A obtains first six) = $\frac{1}{6} + \frac{25}{36} \times \frac{1}{6} + \left(\frac{25}{36}\right)^2 \times \frac{1}{6} + \ldots$ (M1)

recognizing that the common ratio is $\frac{25}{36}$ (A1)

P(A obtains first six) = $\frac{\frac{1}{6}}{1 - \frac{25}{36}}$ (by summing the infinite GP) M1

$= \frac{6}{11}$ A1

[7]

5. (a) $\sin(2n + 1)x \cos x - \cos(2n + 1)x \sin x = \sin(2n + 1)x - x$ M1A1
$= \sin 2nx$ AG

(b) if $n = 1$ M1

LHS = $\cos x$

RHS = $\dfrac{\sin 2x}{2\sin x} = \dfrac{2\sin x\cos x}{2\sin x} = \cos x$ M1

so LHS = RHS and the statement is true for $n = 1$ R1

assume true for $n = k$ M1

Note: Only award M1 if the word true appears.
Do not award M1 for 'let $n = k$' only.
Subsequent marks are independent of this M1.

so $\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x = \dfrac{\sin 2kx}{2\sin x}$

if $n = k + 1$ then

$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x + \cos(2k+1)x$ M1

$= \dfrac{\sin 2kx}{2\sin x} \cos(2k+1)x$ A1

$= \dfrac{\sin 2kx + 2\cos(2k+1)x\sin x}{2\sin x}$ M1

$= \dfrac{\sin(2k+1)x\cos x - \cos(2k+1)x\sin x + 2\cos(2k+1)x\sin x}{2\sin x}$ M1

$= \dfrac{\sin(2k+1)x\cos x + \cos(2k+1)x\sin x}{2\sin x}$ A1

$= \dfrac{\sin(2k+2)x}{2\sin x}$ M1

$= \dfrac{\sin 2(k+1)x}{2\sin x}$ A1

so if true for $n = k$, then also true for $n = k + 1$

as true for $n = 1$ then true for all $n \in \mathbb{Z}^+$ R1

Note: Final R1 is independent of previous work.

(c) $\dfrac{\sin 4x}{2\sin x} = \dfrac{1}{2}$ M1A1

$\sin 4x = \sin x$

$4x = x \Rightarrow x = 0$ but this is impossible

$4x = \pi - x \Rightarrow x = \dfrac{\pi}{5}$ A1

$4x = 2\pi + x \Rightarrow x = \dfrac{2\pi}{3}$ A1

$4x = 3\pi - x \Rightarrow x = \dfrac{3\pi}{5}$ A1

for not including any answers outside the domain R1

Note: Award the first M1A1 for correctly obtaining $8\cos^3 x - 4\cos x - 1 = 0$ or equivalent and subsequent marks as appropriate including the answers $\arccos\left(-\dfrac{1}{2}, \dfrac{1 \pm \sqrt{5}}{4}\right)$.

[20]

6. (a) (i) $1 \times 2 + 2 \times 3 + ... + n(n+1) = \dfrac{1}{3}n(n+1)(n+2)$ R1

(ii) LHS = 40; RHS = 40 A1

(b) the sequence of values are:

5, 7, 11, 19, 35 … or an example A1

35 is not prime, so Bill′s conjecture is false R1AG

(c) $P(n) : 5 \times 7^n + 1$ is divisible by 6
$P(1)$: 36 is divisible by 6 $\Rightarrow$ $P(1)$ true A1
assume $P(k)$ is true $(5 \times 7^k + 1 = 6r)$ M1

Note: Do **not** award M1 for statement starting 'let $n = k$'.
Subsequent marks are independent of this M1.

consider $5 \times 7^{k+1} + 1$ M1
$= 7(6r - 1) + 1$ (A1)
$= 6(7r - 1) \Rightarrow P(k + 1)$ is true A1
$P(1)$ true and $P(k)$ true $\Rightarrow$ $P(k + 1)$ true, so by MI $P(n)$ is true for all $n \in \mathbb{Z}^+$ R1

Note: Only award R1 if there is consideration of P(1), P(k) and P (k + 1) in the final statement.

Only award R1 if at least one of the two preceding A marks has been awarded.

[10]

7. (a) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$ (M1)
$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$ A1

(b) from De Moivre's theorem
$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ (M1)
$\cos 3\theta + i\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$
equating real parts
$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ M1
$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$ A1
$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$
$= 4\cos^3\theta - 3\cos\theta$ AG

Note: Do not award marks if part (a) is not used.

(c) $(\cos\theta + i\sin\theta)^5 =$
$\cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3$
$+ 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$ (A1)
from De Moivre's theorem
$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ M1
$= \cos^5\theta - 10\cos^3\theta(1 - \cos^2\theta) + 5\cos\theta(1 - \cos^2\theta)^2$ A1
$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$
$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ AG

Note: If compound angles used in (b) and (c), then marks can be allocated in (c) only.

(d) $\cos 5\theta + \cos 3\theta + \cos \theta$

$= (16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta) + (4 \cos^3 \theta - 3 \cos \theta) + \cos \theta = 0$ M1

$16 \cos^5 \theta - 16 \cos^3 \theta + 3 \cos\theta = 0$ A1

$\cos \theta\, (16 \cos^4 \theta - 16 \cos^2 \theta + 3) = 0$

$\cos \theta\, (4 \cos^2 \theta - 3)(4 \cos^2 \theta - 1) = 0$ A1

$\therefore \cos \theta = 0; \pm\frac{\sqrt{3}}{2}; \pm\frac{1}{2}$ A1

$\therefore \theta = \pm\frac{\pi}{6}; \pm\frac{\pi}{3}; \pm\frac{\pi}{2}$ A2

(e) $\cos 5\theta = 0$

$5\theta = \ldots\frac{\pi}{2}; \left(\frac{3\pi}{2}; \frac{5\pi}{2}\right); \frac{7\pi}{2}; \ldots$ (M1)

$\theta = \ldots\frac{\pi}{10}; \left(\frac{3\pi}{10}; \frac{5\pi}{10}\right); \frac{7\pi}{10}; \ldots$ (M1)

Note: These marks can be awarded for verifications later in the question.

now consider $16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta = 0$ M1

$\cos \theta\, (16 \cos^4 \theta - 20 \cos^2 \theta + 5) = 0$

$\cos^2\theta = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}$; $\cos \theta = 0$ A1

$\cos \theta = \pm\sqrt{\frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}}$

$\cos\frac{\pi}{10} = \pm\sqrt{\frac{20 + \sqrt{400 - 4(16)(5)}}{32}}$ since max value of cosine $\Rightarrow$ angle closest to zero R1

$\cos\frac{\pi}{10} = \sqrt{\frac{4.5 + 4\sqrt{25 - 4(5)}}{4.8}} = \sqrt{\frac{5 + \sqrt{5}}{8}}$ A1

$\cos\frac{7\pi}{10} = -\sqrt{\frac{5 - \sqrt{5}}{8}}$ A1A1

[22]

8. (a) $\text{AB} = \sqrt{1^2 + (2 - \sqrt{3})^2}$ M1

$= \sqrt{88 - 4\sqrt{3}}$ A1

$= 2\sqrt{2 - \sqrt{3}}$ A1

(b) **METHOD 1**

$\arg z_1 = -\frac{\pi}{4}, \ \arg z_2 = -\frac{\pi}{3}$ A1A1

Note: Allow $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

Note: Allow degrees at this stage.

$\hat{AOB} = \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$ (accept $-\frac{\pi}{12}$) A1

Note: Allow FT for final A1.

METHOD 2

attempt to use scalar product or cosine rule M1

$\cos \hat{AOB} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ A1

$\hat{AOB} = \frac{\pi}{12}$ A1

[6]

9. $u_4 = u_1 + 3d = 7$, $u_9 = u_1 + 8d = 22$ A1A1

Note: $5d = 15$ gains both above marks

$u_1 = –2$, $d = 3$ A1

$S_n = \frac{n}{2}(–4 + (n – 1)3) > 10000$ M1

$n = 83$ A1

[5]

10. (a) let the first three terms of the geometric sequence be given by u_1, $u_1 r$, $u_1 r^2$

$\therefore \ u_1 = a + 2d$, $u_1 r = a + 3d$ and $u_1 r^2 = a + 6d$ (M1)

$\frac{a+6d}{a+3d} = \frac{a+3d}{a+2d}$ A1

$a^2 + 8ad + 12d^2 = a^2 + 6ad + 9d^2$ A1

$2a + 3d = 0$

$a = -\frac{3}{2}d$ AG

(b) $u_1 = \frac{d}{2}, u_1 r = \frac{3d}{2}, \left(u_1 r^2 = \frac{9d}{2}\right)$ M1

$r = 3$ A1

geometric 4th term $u_1r^3 = \dfrac{27d}{2}$ A1

arithmetic 16th term $a + 15d = -\dfrac{3}{2}d + 15d$ M1

$= \dfrac{27d}{2}$ A1

Note: Accept alternative methods.

[8]

11. (a) $\boldsymbol{A}^2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta - \cos\theta\sin\theta & -\sin^2\theta + \cos^2\theta \end{pmatrix}$ M1(A1)

$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{pmatrix}$ A1

$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$ AG

(b) let P(n) be the proposition that $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$

for all $n \in \mathbb{Z}^+$

P(1) is true A1

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^1 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

assume P(k) to be true A1

Note: Must see the word 'true' or equivalent, that makes clear an assumption is being made that P(k) is true.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^k = \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}$$

consider P(k + 1)

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{k+1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^k \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{(M1)}$$

$$= \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{A1}$$

$$= \begin{pmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & \cos k\theta\sin\theta + \sin k\theta\cos\theta \\ -\sin k\theta\cos\theta - \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{pmatrix} \quad \text{A1}$$

$$= \begin{pmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{pmatrix} \quad \text{A1}$$

if P(k) is true then P(k + 1) is true and since P(1) is true then P(n) is true for all $n \in \mathbb{Z}^+$ R1

Note: The final R1 can only be gained if the M1 has been gained.

(c) **EITHER**

$\boldsymbol{A}^{-1} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix}$ from formula

$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ A1

$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

M1

Note: Accept either just $\boldsymbol{A}^{-1}\boldsymbol{A}$ or just $\boldsymbol{A}\boldsymbol{A}^{-1}$.

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ A1

$\therefore \boldsymbol{A}^{-1}$ is inverse of $\boldsymbol{A}$

OR

$\boldsymbol{A}^{-1} = \dfrac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ M1

$\boldsymbol{A}^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ A1

putting $n = -1$ in formula gives inverse A1

[13]

12. (a) (i) $\omega^3 = \left(\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)^3$

$= \cos\left(3\times\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(3\times\frac{2\pi}{3}\right)$ (M1)

$= \cos 2\pi + \mathrm{i}\sin 2\pi$ A1

$= 1$ AG

(ii) $1 + \omega + \omega^2 = 1 + \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right)$ M1A1

$= 1 + -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2} - \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$ A1

$= 0$ AG

(b) (i) $e^{i\theta} + e^{i\left(\theta + \frac{2\pi}{3}\right)} + e^{i\left(\theta + \frac{4\pi}{3}\right)}$

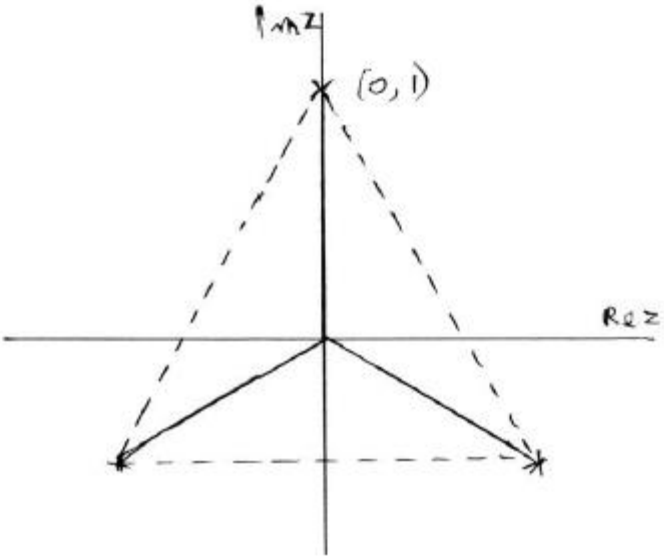
$= e^{i\theta} + e^{i\theta}e^{i\left(\frac{2\pi}{3}\right)} + e^{i\theta}e^{i\left(\frac{4\pi}{3}\right)}$ (M1)

$= \left(e^{i\theta}\left(1 + e^{i\left(\frac{2\pi}{3}\right)} + e^{i\left(\frac{4\pi}{3}\right)}\right)\right)$

$= e^{i\theta}(1 + \omega + \omega^2)$ A1

$= 0$ AG

(ii)



A1A1

Note: Award A1 for one point on the imaginary axis and another point marked with approximately correct modulus and argument. Award A1 for third point marked to form an equilateral triangle centred on the origin.

(c) (i) attempt at the expansion of at least two linear factors (M1)

$(z - 1)z^2 - z(\omega + \omega^2) + \omega^3$ or equivalent (A1)

use of earlier result (M1)

$F(z) = (z - 1)(z^2 + z + 1) = z^3 - 1$ A1

(ii) equation to solve is $z^3 = 8$ (M1)

$z = 2, 2\omega, 2\omega^2$ A2

Note: Award A1 for 2 correct solutions.

[16]

13. (a) using the factor theorem $z + 1$ is a factor (M1)

$z^3 + 1 = (z + 1)(z^2 - z + 1)$ A1

(b) (i) **METHOD 1**

$z^3 = -1 \Rightarrow z^3 + 1 = (z+1)(z^2 - z + 1) = 0$ (M1)

solving $z^2 - z + 1 = 0$ M1

$z = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm \mathrm{i}\sqrt{3}}{2}$ A1

therefore one cube root of -1 is γ AG

METHOD 2

$\gamma^2 = \left(\dfrac{1+\mathrm{i}\sqrt{3}}{2}\right)^2 = \dfrac{-1+\mathrm{i}\sqrt{3}}{2}$ M1A1

$\gamma^3 = \dfrac{-1+\mathrm{i}\sqrt{3}}{2} \times \dfrac{-1+\mathrm{i}\sqrt{3}}{2} = \dfrac{-1-3}{4}$ A1

$= -1$ AG

METHOD 3

$\gamma = \dfrac{-1+\mathrm{i}\sqrt{3}}{2} = e^{\mathrm{i}\frac{\pi}{3}}$ A1

$\gamma^3 = e^{\mathrm{i}\pi} - 1$ A1

(ii) **METHOD 1**

as γ is a root of $z^2 - z + 1 = 0$ then $\gamma^2 - \gamma + 1 = 0$ M1R1

$\therefore \gamma^2 = \gamma - 1$ AG

Note: Award M1 for the use of $z^2 - z + 1 = 0$ in any way.
Award R1 for a correct reasoned approach.

METHOD 2

$\gamma^2 = \dfrac{-1+\mathrm{i}\sqrt{3}}{2}$ M1

$\gamma - 1 = \dfrac{1+\mathrm{i}\sqrt{3}}{2} - 1 = \dfrac{-1+\mathrm{i}\sqrt{3}}{2}$ A1

(iii) **METHOD 1**

$(1-\gamma)^6 = (-\gamma^2)^6$ (M1)

$= (\gamma)^{12}$ A1

$= (\gamma^3)^4$ (M1)

$= (-1)^4$

$= 1$ A1

METHOD 2

$(1-\gamma)^6$

$= 1 - 6\gamma + 15\gamma^2 - 20\gamma^3 + 15\gamma^4 - 6\gamma^5 + \gamma^6$ M1A1

Note: Award M1 for attempt at binomial expansion.

use of any previous result e.g. $= 1 - 6\gamma + 15\gamma^2 + 20 - 15\gamma + 6\gamma^2 + 1$ M1

$= 1$ A1

Note: As the question uses the word 'hence', other methods that do not use previous results are awarded no marks.

(c) **METHOD 1**

$$\boldsymbol{A}^2 = \begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix}\begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix} = \begin{pmatrix} \gamma^2 & \gamma + \frac{1}{\gamma} \\ 0 & \frac{1}{\gamma^2} \end{pmatrix}$$ A1

$$\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I} = \begin{pmatrix} \gamma^2 - \gamma + 1 & \gamma + \frac{1}{\gamma} - 1 \\ 0 & \frac{1}{\gamma^2} - \frac{1}{\gamma} + 1 \end{pmatrix}$$ M1

from part (b)

$\gamma^2 - \gamma + 1 = 0$

$\gamma + \frac{1}{\gamma} - 1 = \frac{1}{\gamma}(\gamma^2 - \gamma + 1) = 0$ A1

$\frac{1}{\gamma^2} + \frac{1}{\gamma} + 1 = \frac{1}{\gamma^2}(\gamma^2 - \gamma + 1) = 0$ A1

hence $\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I} = 0$ AG

METHOD 2

$$\boldsymbol{A}^2 = \begin{pmatrix} \frac{-1+\mathrm{i}\sqrt{3}}{2} & 1 \\ 0 & \frac{-1-\mathrm{i}\sqrt{3}}{2} \end{pmatrix}$$ A1A1A1

Note: Award 1 mark for each of the non-zero elements expressed in this form.

verifying $\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I} = 0$

(d) (i) $\boldsymbol{A}^2 = \boldsymbol{A} - \boldsymbol{I}$

$\Rightarrow \boldsymbol{A}^3 = \boldsymbol{A}^2 - \boldsymbol{A}$ M1A1

$= \boldsymbol{A} - \boldsymbol{I} - \boldsymbol{A}$ A1

$= -\boldsymbol{I}$ AG

Note: Allow other valid methods.

(ii) $\boldsymbol{I} = \boldsymbol{A} - \boldsymbol{A}^2$

$\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} - \boldsymbol{A}^{-1}\boldsymbol{A}^2$ M1A1

$\Rightarrow \boldsymbol{A}^{-1} = \boldsymbol{I} - \boldsymbol{A}$ AG

Note: Allow other valid methods.

[20]

14. (a) $2x^2 + x - 3 = (2x + 3)(x - 1)$ A1

Note: Accept $2\left(x+\frac{3}{2}\right)(x-1)$.

Note: Either of these may be seen in (b) and if so A1 should be awarded.

(b) **EITHER**

$(2x^2 + x - 3)^8 = (2x + 3)^8(x - 1)^8$ M1

$= (3^8 + 8(3^7)(2x) + \ldots)((-1)^8 + 8(-1)^7(x) + \ldots)$ (A1)

coefficient of $x = 3^8 \times 8 \times (-1)^7 + 3^7 \times 8 \times 2 \times (-1)^8$ M1

$= -17\ 496$ A1

Note: Under FT, final A1 can only be achieved for an integer answer.

OR

$(2x^2 + x - 3)^8 = (3 - (x - 2x^2))^8$ M1

$= 3^8 + 8(-(x - 2x^2)(3^7) + \ldots)$ (A1)

coefficient of $x = 8 \times (-1) \times 3^7$ M1

$= -17\ 496$ A1

Note: Under FT, final A1 can only be achieved for an integer answer.

[5]

15. $\log_{x+1} y = 2$

$\log_{y+1} x = \frac{1}{4}$

so $(x + 1)^2 = y$ A1

$(y+1)^{\frac{1}{4}} = x$ A1

EITHER

$x^4 - 1 = (x + 1)^2$ M1
$x = -1$, not possible R1
$x = 1.70$, $y = 7.27$ A1A1

OR

$(x^2 + 2x + 2)^{\frac{1}{4}} - x = 0$ M1
attempt to solve or graph of LHS M1
$x = 1.70$, $y = 7.27$ A1A1

[6]

16. (a) $18n - 10$ (or equivalent) A1

(b) $\sum_{1}^{n}(18r - 10)$ (or equivalent) A1

(c) by use of GDC or algebraic summation or sum of an AP (M1)
$\sum_{1}^{15}(18r - 10) = 2010$ A1

[4]

17. (a) the expression is
$\frac{n!}{(n-3)!3!} - \frac{(2n)!}{(2n-2)!2!}$ (A1)
$\frac{n(n-1)(n-2)}{6} - \frac{2n(2n-1)}{2}$ M1A1
$= \frac{n(n^2 - 15n + 8)}{6} \left(= \frac{n^3 - 15n + 8n)}{6}\right)$ A1

(b) the inequality is
$\frac{n^3 - 15n^2 + 8n}{6} > 32n$
attempt to solve cubic inequality or equation (M1)
$n^3 - 15n^2 - 184n > 0$ $n(n - 23)(n + 8) > 0$
$n > 23$ $(n \geq 24)$ A1

[6]

18. (a) $z^3 = 2\sqrt{2}e^{\frac{3\pi i}{4}}$ (M1)(A1)

$z_1 = \sqrt{2}e^{\frac{\pi i}{4}}$ A1

adding or subtracting $\frac{2\pi i}{3}$ M1

$z_2 = \sqrt{2}e^{\frac{\pi i}{4}+\frac{2\pi i}{3}} = \sqrt{2}e^{\frac{11\pi i}{12}}$ A1

$z_3 = \sqrt{2}e^{\frac{\pi i}{4}-\frac{2\pi i}{3}} = \sqrt{2}e^{-\frac{5\pi i}{12}}$ A1

Notes: Accept equivalent solutions e.g. $z_3 = \sqrt{2}e^{\frac{19\pi i}{12}}$

Award marks as appropriate for solving $(a + bi)^3 = -2 + 2i$.

Accept answers in degrees.

(b) $\sqrt{2}e^{\frac{\pi i}{4}}\left(= \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right)$ A1

$= 1 + i$ AG

Note: Accept geometrical reasoning.

[7]

19. (a) using de Moivre's theorem

$z^n + \frac{1}{z^n} = \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta$ (= $2 \cos n\theta$), imaginary part of which is 0 M1A1

so $\text{Im}\left(z^n + \frac{1}{z^n}\right) = 0$ AG

(b) $\dfrac{z-1}{z+1}=\dfrac{\cos\theta+\mathrm{i}\sin\theta-1}{\cos\theta+\mathrm{i}\sin\theta+1}$

$=\dfrac{(\cos\theta-1+\mathrm{i}\sin\theta)(\cos\theta+1-\mathrm{i}\sin\theta)}{(\cos\theta+1+\mathrm{i}\sin\theta)(\cos\theta+1-\mathrm{i}\sin\theta)}$ M1A1

Note: Award M1 for an attempt to multiply numerator and denominator by the complex conjugate of their denominator.

$\Rightarrow \mathrm{Re}\left(\dfrac{z-1}{z+1}\right)=\dfrac{(\cos\theta-1)(\cos\theta+1)+\sin^2\theta}{\text{real denominator}}$ M1A1

Note: Award M1 for multiplying out the numerator.

$\dfrac{\cos^2\theta+\sin^2\theta-1}{\text{real denominator}}$ A1

= 0 AG

[7]

20. $\left(x^2-\dfrac{2}{x}\right)^4=(x^2)^4+4(x^2)^3\left(-\dfrac{2}{x}\right)+6(x^2)^2\left(-\dfrac{2}{x}\right)^2+4(x^2)\left(-\dfrac{2}{x}\right)^3+\left(-\dfrac{2}{x}\right)^4$ (M1)

$= x^8 - 8x^5 + 24x^2 - \dfrac{32}{x}+\dfrac{16}{x^4}$ A3

Note: Deduct one *A* mark for each incorrect or omitted term.

[4]

21. (100 + 101 + 102 + ... + 999) – (102 + 105 + ... + 999) (M1)

$= \dfrac{900}{2}(100+999)-\dfrac{300}{2}(102+999)$ M1A1A1

= 329 400 A1 N5

Note: A variety of other acceptable methods may be seen including for example $\dfrac{300}{2}(201+1995)$ or $\dfrac{600}{2}(100+998)$.

[5]

22. (a) $|z| = \sqrt{5}$ and $|w| = \sqrt{4+a^2}$

$|w| = 2|z|$

$\sqrt{4+a^2}=2\sqrt{5}$

attempt to solve equation M1

Note: Award M0 if modulus is not used.

$a = \pm 4$ A1A1 N0

(b) $zw = (2 - 2a) + (4 + a)\mathrm{i}$ A1

forming equation $2 - 2a = 2(4 + a)$ M1

$a = -\frac{3}{2}$ A1 N0

[6]

23. **METHOD 1**

(a) $u_n = S_n - S_{n-1}$ (M1)

$= \frac{7^n - a^n}{7^n} - \frac{7^{n-1} - a^{n-1}}{7^{n-1}}$ A1

(b) **EITHER**

$u_1 = 1 - \frac{a}{7}$ A1

$u_2 = 1 - \frac{a^2}{7^2} - \left(1 - \frac{a}{7}\right)$ M1

$= \frac{a}{7}\left(1 - \frac{a}{7}\right)$ A1

common ratio $= \frac{a}{7}$ A1

OR

$u_n = 1 - \left(\frac{a}{7}\right)^n - 1 + \left(\frac{a}{7}\right)^{n-1}$ M1

$= \left(\frac{a}{7}\right)^{n-1}\left(1 - \frac{a}{7}\right)$

$= \frac{7-a}{7}\left(\frac{a}{7}\right)^{n-1}$ A1

$u_1 = \frac{7-a}{7}$, common ratio $= \frac{a}{7}$ A1A1

(c) (i) $0 < a < 7$ (accept $a < 7$) A1

(ii) 1 A1

METHOD 2

(a) $u_n = br^{n-1} = \left(\frac{7-a}{7}\right)\left(\frac{a}{7}\right)^{n-1}$ A1A1

(b) for a GP with first term b and common ratio r

$S_n = \frac{b(1-r^n)}{1-r} = \left(\frac{b}{1-r}\right) - \left(\frac{b}{1-r}\right)r^n$ M1

as $S_n = \frac{7^n - a^n}{7^n} = 1 - \left(\frac{a}{7}\right)^n$

comparing both expressions M1

$\frac{b}{1-r} = 1$ and $r = \frac{a}{7}$

$b = 1 - \frac{a}{7} = \frac{7-a}{7}$

$u_1 = b = \frac{7-a}{7}$, common ratio $= r = \frac{a}{7}$ A1A1

Note: Award method marks if the expressions for b and r are deduced in part (a).

(c) (i) $0 < a < 7$ (accept $a < 7$) A1

(ii) 1 A1

[8]

24. METHOD 1

(a) $|\boldsymbol{a} - \boldsymbol{b}| = \sqrt{|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\alpha}$ M1

$= \sqrt{2 - 2\cos\alpha}$ A1

$|\boldsymbol{a} + \boldsymbol{b}| = \sqrt{|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos(\pi - \alpha)}$

$= \sqrt{2 + 2\cos\alpha}$ A1

Note: Accept the use of a, b for $|\boldsymbol{a}|$, $|\boldsymbol{b}|$.

(b) $\sqrt{2 + 2\cos\alpha} = 3\sqrt{2 - 2\cos\alpha}$ M1

$\cos\alpha = \frac{4}{5}$ A1

METHOD 2

(a) $|\boldsymbol{a} - \boldsymbol{b}| = 2\sin\frac{\alpha}{2}$ M1A1

$|\boldsymbol{a} + \boldsymbol{b}| = 2\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = 2\cos\frac{\alpha}{2}$ A1

Note: Accept the use of a, b for $|\boldsymbol{a}|$, $|\boldsymbol{b}|$.

(b) $2\cos\frac{\alpha}{2} = 6\sin\frac{\alpha}{2}$

$\tan\frac{\alpha}{2} = \frac{1}{3} \Rightarrow \cos^2\frac{\alpha}{2} = \frac{9}{10}$ M1

$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = \frac{4}{5}$ A1

[5]

25. METHOD 1

1 + i is a zero $\Rightarrow$ 1 – i is a zero (A1)

1 – 2i is a zero $\Rightarrow$ 1 + 2i is a zero (A1)

$(x-(1-i))(x-(1+i)) = (x^2-2x+2)$ (M1)A1

$(x-(1-2i))(x-(1+2i)) = (x^2-2x+5)$ A1

$p(x) = (x^2-2x+2)(x^2-2x+5)$ M1

$\qquad = x^4-4x^3+11x^2-14x+10$ A1

$a = -4,\ b = 11,\ c = -14,\ d = 10$

METHOD 2

$p(1+i) = -4 + (-2+2i)a + (2i)b + (1+i)c + d$ M1

$p(1+i) = 0 \Rightarrow \begin{cases} -4-2a+c+d=0 \\ 2a+2b+c=0 \end{cases}$ M1A1A1

$p(1-2i) = -7 + 24i + (-11+2i)a + (-3-4i)b + (1-2i)c + d$

$p(1-2i) = 0 \Rightarrow \begin{cases} -7-11a-3b+c+d=0 \\ 24+2a-4b-2c=0 \end{cases}$ A1

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 \\ -11 & -3 & 1 & 1 \\ 2 & -4 & -2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \\ 7 \\ -24 \end{pmatrix} = \begin{pmatrix} -4 \\ 11 \\ -14 \\ 10 \end{pmatrix}$$ M1A1

$a = -4,\ b = 11,\ c = -14,\ d = 10$

[7]

26. (a) **METHOD 1**

$V = a^3 - \frac{1}{a^3}$ A1

$x^3 = \left(a - \frac{1}{a}\right)^3$ M1

$= a^3 - 3a + \frac{3}{a} - \frac{1}{a^3}$

$= a^3 - \frac{1}{a^3} - 3\left(a - \frac{1}{a}\right)$ (or equivalent) (A1)

$\Rightarrow a^3 - \frac{1}{a^3} = x^3 + 3x$

$V = x^3 + 3x$ A1 N0

METHOD 2

$V = a^3 - \frac{1}{a^3}$ A1

attempt to use difference of cubes formula, $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ M1

$V = \left(a - \frac{1}{a}\right)\left(a^2 + 1 + \left(\frac{1}{a}\right)^2\right)$

$= \left(a - \frac{1}{a}\right)\left(\left(a - \frac{1}{a}\right)^2 + 3\right)$ (A1)

$= x(x^2 + 3)$ or $x^3 + 3x$ A1 N0

METHOD 3

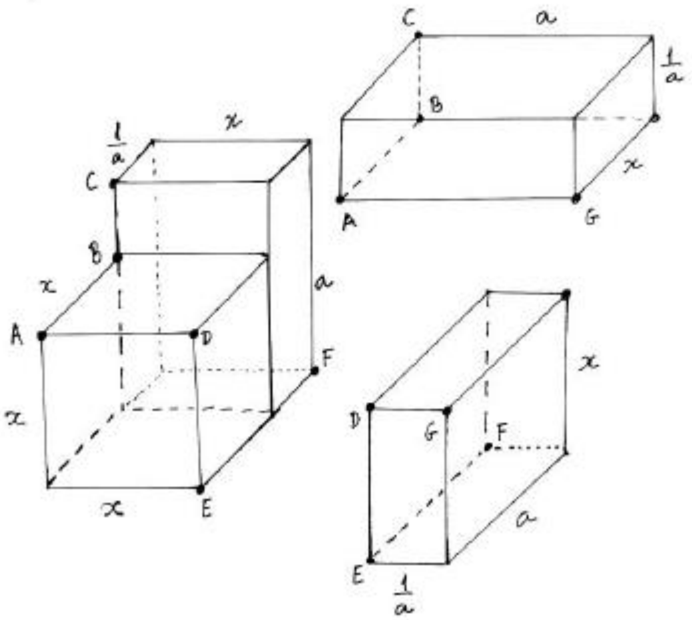


diagram showing that the solid can be decomposed	M1	
into three congruent $x \times a \times \frac{1}{a}$ cuboids with volume x	A1	
and a cube with edge x with volume x^3	A1	
so, $V = x^3 + 3x$	A1	N0

(b) **Note:** Do not accept any method where candidate substitutes the given value of a into $x = a - \frac{1}{a}$.

METHOD 1

$V = 4x \Leftrightarrow x^3 + 3x = 4x \Leftrightarrow x^3 - x = 0$		
$\Leftrightarrow x(x-1)(x+1) = 0$	M1	
$\Rightarrow x = 1$ as $x > 0$	A1	
so, $a - \frac{1}{a} = 1 \Rightarrow a^2 - a - 1 = 0 \Rightarrow a = \frac{1 \pm \sqrt{5}}{2}$	M1A1	
as $a > 1$, $a = \frac{1+\sqrt{5}}{2}$	AG	N0

METHOD 2

$a^3 - \frac{1}{a^3} = 4\left(a - \frac{1}{a}\right) \Rightarrow = a^6 - 4a^4 + 4a^2 - 1 = 0$

$\Leftrightarrow (a^2 - 1)(a^4 - 3a^2 + 1) = 0$ M1A1

as $a > 1 \Rightarrow a^2 > 1, a^2 = \frac{3+\sqrt{5}}{2} \Leftrightarrow a^2 = \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2}$ M1A1

$\Rightarrow a = \frac{1+\sqrt{5}}{2}$ AG N0

[8]

27. (a) (i) $a, 2a, 3a, ..., na$ are n consecutive terms of an AP with first term a and common difference a

so their mean is $\frac{a+2a+3a+...+na}{n} = \frac{a\frac{n(n+1)}{2}}{n}$ M1A1

$= \frac{a(n+1)}{2}$ AG N0

(ii) $4 + 2 \times 4 + 3 \times 4 + ... + 4n > \frac{4(n+1)}{2} + 100$ M1

$\frac{4n(n+1)}{2} > 2(n + 1) + 100$ A1

$2n^2 + 2n > 2n + 102$

attempt to solve (M1)

$n^2 > 51$

so the minimum value of n that satisfies the condition is 8 A1 N0

Note: Award M1A1(M1)A1 for use of equations if there is a clear conversion to an inequality.

(b) (i) $M = \frac{x_1 + ... + x_m + y_1 + ... + y_n}{m+n}$ M1

$= \frac{0 \times m + 1 \times n}{m+n}$ A1

$= \frac{n}{m+n}$ AG N0

EITHER

$$S = \sqrt{\frac{\left(0-\frac{n}{m+n}\right)^2 \times m + \left(1-\frac{n}{m+n}\right)^2 \times n}{m+n}}$$ M1A1

attempt to simplify

$$S = \sqrt{\frac{\frac{m^2n+n^2m}{(m+n)^2}}{m+n}} = \sqrt{\frac{mn(m+n)}{(m+n)^3}}$$

$$= \sqrt{\frac{mn}{(m+n^2)}}$$ A1

$$= \frac{\sqrt{mn}}{m+n}$$ AG N0

OR

$$\text{Var}(x) = \frac{\sum_{i=1}^{m} x_i^2 + \sum_{i=1}^{n} y_i^2}{m+n} - M^2$$ M1A1

attempt to simplify M1

$$\text{Var}(x) = \frac{n}{m+n} - \frac{n^2}{(m+n)^2}$$

$$= \frac{n}{m+n}\left(1-\frac{n}{m+n}\right)$$

$$= \frac{n}{m+n} \times \frac{m}{m+n}$$

$$= \frac{mn}{(m+n)^2}$$ A1

$$\therefore S = \frac{\sqrt{mn}}{m+n}$$ AG N0

(ii) $M = S \Rightarrow \frac{n}{m+n} = \frac{\sqrt{mn}}{m+n}$ A1

attempt to solve M1

$\Rightarrow n = \sqrt{mn}$

$\Rightarrow n = m$, as $n > 0$ A1

so, then the set has $2n$ numbers, $x_1, \ldots, x_n, y_1, \ldots, y_n$

from which the first n are all 0 and the last n are all 1 (M1)

hence the value of the median is $\frac{x_n + y_1}{2} = \frac{1}{2}$ A1 N0

[17]

28. (a) $|z| = z$, $\arg(z) = 0$ A1A1

so $L(z) = \ln z$ AG N0

(b) (i) $L(-1) = \ln 1 + \mathrm{i}\pi = \mathrm{i}\pi$ A1A1 N2

(ii) $L(1 - \mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\frac{7\pi}{4}$ A1A1 N2

(iii) $L(-1 + \mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\frac{3\pi}{4}$ A1 N1

(c) for comparing the product of two of the above results with the third M1

for stating the result $-1 + \mathrm{i} = -1 \times (1 - \mathrm{i})$ and $L(-1 + \mathrm{i}) \neq L(-1) + L(1 - \mathrm{i})$ R1

hence, the property $L(z_1 z_2) = L(z_1) + L(z_2)$

does not hold for all values of z_1 and z_2 AG N0

[9]

29. (a) **METHOD 1**

$\frac{z+\mathrm{i}}{z+2} = \mathrm{i}$

$z + \mathrm{i} = \mathrm{i}z + 2\mathrm{i}$ M1

$(1 - \mathrm{i})z = \mathrm{i}$ A1

$z = \frac{\mathrm{i}}{1-\mathrm{i}}$ A1

EITHER

$z = \frac{\mathrm{cis}\left(\frac{\pi}{2}\right)}{\sqrt{2}\mathrm{cis}\left(\frac{3\pi}{4}\right)}$ M1

$z = \frac{\sqrt{2}}{2}\mathrm{cis}\left(\frac{3\pi}{4}\right) \left(\text{or } \frac{1}{\sqrt{2}}\mathrm{cis}\left(\frac{3\pi}{4}\right)\right)$ A1A1

OR

$z = \frac{-1+\mathrm{i}}{2} \left(= -\frac{1}{2} + \frac{1}{2}\mathrm{i}\right)$ M1

$z = \frac{\sqrt{2}}{2}\mathrm{cis}\left(\frac{3\pi}{4}\right)\left(\text{or } \frac{1}{\sqrt{2}}\mathrm{cis}\left(\frac{3\pi}{4}\right)\right)$ A1A1

METHOD 2

$\mathrm{i} = \frac{x+\mathrm{i}(y+1)}{x+2+\mathrm{i}y}$ M1

$x + \mathrm{i}(y + 1) = -y + \mathrm{i}(x + 2)$ A1

$x = -y$; $x + 2 = y + 1$ A1

solving, $x = -\frac{1}{2}$; $y = \frac{1}{2}$ A1

$z = -\frac{1}{2} + \frac{1}{2}\mathrm{i}$

$z = \frac{\sqrt{2}}{2}\mathrm{cis}\left(\frac{3\pi}{4}\right) \left(\text{or } \frac{1}{\sqrt{2}}\mathrm{cis}\left(\frac{3\pi}{4}\right)\right)$ A1A1

Note: Award A1 fort the correct modulus and A1 for the correct argument, but the final answer must be in the form r cis θ.
Accept 135° for the argument.

(b) substituting $z = x + \mathrm{i}y$ to obtain $w = \dfrac{x+(y+1)\mathrm{i}}{(x+2)+y\mathrm{i}}$ (A1)

use of $(x + 2) - y\mathrm{i}$ to rationalize the denominator M1

$\omega = \dfrac{x(x+2)+y(y+1)+\mathrm{i}(-xy+(y+1)(x+2))}{(x+2)^2+y^2}$ A1

$= \dfrac{(x^2+2x+y^2+y)+\mathrm{i}(x+2y+2)}{(x+2)^2+y^2}$ AG

(c) $\mathrm{Re}\,\omega = \dfrac{x^2+2x+y^2+y}{(x+2)^2+y^2} = 1$ M1

$\Rightarrow x^2 + 2x + y^2 + y = x^2 + 4x + 4 + y^2$ A1

$\Rightarrow y = 2x + 4$ A1

which has gradient $m = 2$ A1

(d) **EITHER**

$\arg(z) = \dfrac{\pi}{4} \Rightarrow x = y$ (and $x, y > 0$) (A1)

$\omega = \dfrac{2x^2+3x}{(x+2)^2+x^2} + \dfrac{\mathrm{i}(3x+2)}{(x+2)^2+x^2}$

if $\arg(\omega) = \theta \Rightarrow \tan\theta = \dfrac{3x+2}{2x^2+3x}$ (M1)

$\dfrac{3x+2}{2x^2+3x} = 1$ M1A1

OR

$\arg(z) = \dfrac{\pi}{4} \Rightarrow x = y$ (and $x, y > 0$) A1

$\arg(w) = \dfrac{\pi}{4} \Rightarrow x^2 + 2x + y^2 + y = x + 2y + 2$ M1

solve simultaneously M1

$x^2 + 2x + x^2 + x = x + 2x + 2$ (or equivalent) A1

THEN

$x^2 = 1$

$x = 1$ (as $x > 0$) A1

Note: Award A0 for $x = \pm 1$.

$|z| = \sqrt{2}$ A1

Note: A1low FT from incorrect values of x.

[19]

30. **METHOD 1**

$5(2a + 9d) = 60$ (or $2a + 9d = 12$) M1A1

$10(2a + 19d) = 320$ (or $2a + 19d = 32$) A1 solve simultaneously to obtain M1

$a = -3$, $d = 2$ A1

the 15th term is $-3 + 14 \times 2 = 25$ A1

Note: FT the final A1 on the values found in the penultimate line.

METHOD 2

with an AP the mean of an even number of consecutive terms equals the mean of the middle terms (M1)

$\frac{a_{10} + a_{11}}{2} = 16$ (or $a_{10} + a_{11} = 32$) A1

$\frac{a_5 + a_6}{2} = 6$ (or $a_5 + a_6 = 12$) A1

$a_{10} - a_5 + a_{11} - a_6 = 20$ M1

$5d + 5d = 20$

$d = 2$ and $a = -3$ (or $a_5 = 5$ or $a_{10} = 15$) A1

the 15th term is $-3 + 14 \times 2 = 25$ (or $5 + 10 \times 2 = 25$ or $15 + 5 \times 2 = 25$) A1

Note: FT the final A1 on the values found in the penultimate line.

[6]

31. (a) There are 3! ways of arranging the Mathematics books, 5! ways of arranging the English books and 4! ways of arranging the Science books. (A1)

Then we have 4 types of books which can be arranged in 4! ways. (A1)

$3! \times 5! \times 4 \times 4! = 414\,720$ (M1)A1

(b) There are 3! ways of arranging the subject books, and for each of these there are 2 ways of putting the dictionary next to the Mathematics books. (M1)(A1)

$3! \times 5! \times 4! \times 3! \times 2 = 207\,360$ A1

[7]

32. (a) $1 - i\sqrt{3}$ A1

(b) **EITHER**

$(z - (1 + i\sqrt{3}))(z - (1 - i\sqrt{3})) = z^2 - 2z + 4$ (M1)A1

$p(z) = (z - 2)(z^2 - 2z + 4)$ (M1)

$= z^3 - 4z^2 + 8z - 8$ A1

therefore $b = -4$, $c = 8$, $d = -8$

OR

relating coefficients of cubic equations to roots

$-b = 2 + 1 + i\sqrt{3} + 1 - i\sqrt{3} = 4$ M1

$c = 2(1 + i\sqrt{3}) + 2(1 - i\sqrt{3}) + (1 + i\sqrt{3})(1 - i\sqrt{3}) = 8$

$-d = 2(1 + i\sqrt{3})(1 - i\sqrt{3}) = 8$

$b = -4$, $c = 8$, $d = -8$ A1A1A1

(c) $z_2 = 2e^{\frac{i\pi}{3}}$, $z_3 = 2e^{-\frac{i\pi}{3}}$ A1A1A1

Note: Award A1 for modulus,
A1 for each argument.

[8]

33. let $n = 1$

LHS $= 1 \times 1! = 1$

RHS $= (1 + 1)! - 1 = 2 - 1 = 1$

hence true for $n = 1$ R1

assume true for $n = k$

$$\sum_{r=1}^{k} r(r!) = (k + 1)! - 1$$ M1

$$\sum_{r=1}^{k+1} r(r!) = (k + 1)! - 1 + (k + 1) \times (k + 1)!$$ M1A1

$= (k + 1)!(1 + k + 1) - 1$

$= (k + 1)!(k + 2) - 1$ A1

$= (k + 2)! - 1$ A1

hence if true for $n = k$, true for $n = k + 1$ R1

since the result is true for $n = 1$ and $P(k) \Rightarrow P(k + 1)$ the result is proved by mathematical induction $\forall n \in \mathbb{Z}^+$ R1

[8]

34. (a) any appropriate form, e.g. $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ A1

(b) $z^n = \cos n\theta + \mathrm{i}\sin n\theta$ A1

$\frac{1}{z^n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ (M1)

$= \cos n\theta - \mathrm{i}\sin(n\theta)$ A1

therefore $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$ AG

(c) $\left(z-\frac{1}{z}\right)^5 = z^5 + \binom{5}{1}z^4\left(-\frac{1}{z}\right) + \binom{5}{2}z^3\left(-\frac{1}{z}\right)^2 + \binom{5}{3}z^2\left(-\frac{1}{z}\right)^3 + \binom{5}{4}z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$

(M1)(A1)

$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$ A1

(d) $\left(z-\frac{1}{z}\right)^5 = z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$ M1A1

$(2\mathrm{i}\sin\theta)^5 = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta$ M1A1

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$ AG

(e) $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

LHS $= 16\left(\sin\frac{\pi}{4}\right)^5$

$= 16\left(\frac{\sqrt{2}}{2}\right)^5$

$= 2\sqrt{2}\ \left(= \frac{4}{\sqrt{2}}\right)$ A1

RHS $= \sin\left(\frac{5\pi}{4}\right) - 5\sin\left(\frac{3\pi}{4}\right) + 10\sin\left(\frac{\pi}{4}\right)$

$= -\frac{\sqrt{2}}{2} - 5\left(\frac{\sqrt{2}}{2}\right) + 10\left(\frac{\sqrt{2}}{2}\right)$ M1A1

Note: Award M1 for attempted substitution.

$= 2\sqrt{2}\ \left(= \frac{4}{\sqrt{2}}\right)$ A1

hence this is true for $\theta = \frac{\pi}{4}$ AG

(f) $\int_0^{\frac{\pi}{2}} \sin^5 \theta \mathrm{d}\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\mathrm{d}\theta$ M1

$= \frac{1}{16}\left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_0^{\frac{\pi}{2}}$ A1

$= \frac{1}{16}\left[0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right]$ A1

$= \frac{8}{15}$ A1

(g) $\int_0^{\frac{\pi}{2}} \cos^5 \theta \mathrm{d}\theta = \frac{8}{15}$, with appropriate reference to symmetry and graphs. A1R1R1

Note: Award first R1 for partially correct reasoning e.g. sketches of graphs of sin and cos.
Award second R1 for fully correct reasoning involving $\sin^5$ and $\cos^5$.

[22]

35. (a) $i^4 - 5i^3 + 7i^2 - 5i + 6 = 1 + 5i - 7 - 5i + 6$ M1A1
$= 0$ AG N0

(b) i root $\Rightarrow$ –i is second root (M1)A1
moreover, $x^4 - 5x^3 + 7x^2 - 5x + 6 = (x - i)(x + i)\, q(x)$
where $q(x) = x^2 - 5x + 6$
finding roots of $q(x)$
the other two roots are 2 and 3 A1A1

Note: Final A1A1 is independent of previous work.

[6]

36. (a) (i) $\boldsymbol{X} = \boldsymbol{B} - \boldsymbol{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ A1

$\boldsymbol{Y} = \boldsymbol{B}^{-1} - \boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ A1

(ii) $X^{-1} + Y^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ (A1)

$X^{-1} + Y^{-1}$ has no inverse A1

as $\det(X^{-1} + Y^{-1}) = 0$ R1

(b) if $P(n)$: $A^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$

for $n = 1$, $P(1) : A = \begin{pmatrix} 1 & 1 & \frac{1(1+1)}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow P(1)$ is true A1

assume $P(k)$ is true *i.e.* $A^k = \begin{pmatrix} 1 & k & \frac{k(k+1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$ M1

for $n = k + 1$,

$A^{k+1} = A^k A$ or AA^k M1

$= \begin{pmatrix} 1 & k & \frac{k(k+1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1+k & 1+k+\frac{k(k+1)}{2} \\ 0 & 1 & 1+k \\ 0 & 0 & 1 \end{pmatrix}$ M1A1

$= \begin{pmatrix} 1 & 1+k & \frac{(k+1)(k+2)}{2} \\ 0 & 1 & 1+k \\ 0 & 0 & 1 \end{pmatrix}$ A1

hence $P(k) \Rightarrow P(k+1)$ and $P(1)$ is true, so $P(n)$ is true for all $n \in \mathbb{Z}^+$. R1 N0

(c) (i) $A^n(A^n)^{-1} = I \Rightarrow \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ M1

$\Rightarrow \begin{pmatrix} 1 & x+n & y+nx+\frac{n(n+1)}{2} \\ 0 & 1 & x+n \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ A1

solve simultaneous equations to obtain

$x + n = 0$ and $y + nx + \frac{n(n+1)}{2} = 0$ M1

$x = -n$ and $y = \frac{n(n-1)}{2}$ A1A1 N2

(ii) $A^n + (A^n)^{-1} = \Rightarrow \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -n & \frac{n(n-1)}{2} \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & n^2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

A1

[18]

37. EITHER

with no restrictions six people can be seated in 5! = 120 ways A1
we now count the number of ways in which the two restricted people will be sitting next to each other
call the two restricted people p_1 and p_2
they sit next to each other in two ways A1

the remaining people can then be seated in 4! ways A1
the six may be seated (p_1 and p_2 next to each other) in 2 × 4! = 48 ways M1
∴ with p_1 and p_2 not next to each other the number of ways = 120 – 48 = 72 A1 N3

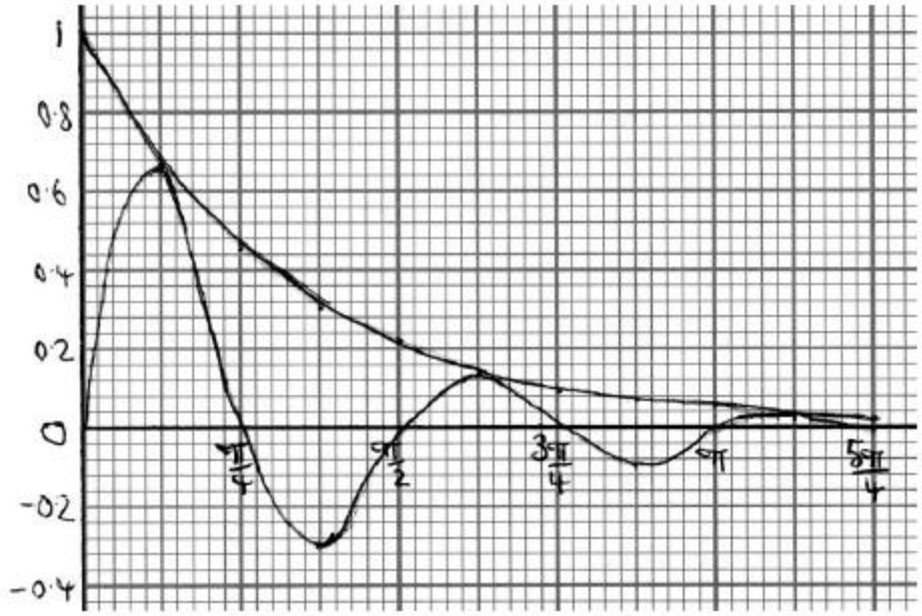
OR

person p_1 seated at table in 1 way A1
p_2 then sits in any of 3 seats (not next to p_1) M1A1
the remaining 4 people can then be seated in 4! ways A1
∴ number ways with p_1 not next to p_2 = 3 × 4! = 72 ways A1 N3

Note: If candidate starts with 6! instead of 5!, potentially leading to an answer of 432, do not penalise.

[5]

38. (a)



A3

Note: Award A1 for each correct **shape**,
A1 for correct relative position.

(b) $e^{-x} \sin(4x) = 0$ (M1)

$\sin(4x) = 0$ A1

$4x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$ A1

$x = 0, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}, \frac{5\pi}{4}$ AG

(c) $e^{-x} = e^{-x} \sin(4x)$ or reference to graph

$\sin 4x = 1$ M1

$4x = \frac{\pi}{4}, \frac{5\pi}{2}, \frac{9\pi}{2}$ A1

$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$ A1 N3

(d) (i) $y = e^{-x} \sin 4x$

$\frac{dy}{dx} = -e^{-x} \sin 4x + 4e^{-x} \cos 4x$ M1A1

$y = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$ A1

verifying equality of gradients at one point R1

verifying at the other two R1

(ii) since $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ at these points they cannot be local maxima R1

(e) (i) maximum when $y' = 4\mathrm{e}^{-x}\cos 4x - \mathrm{e}^{-x}\sin 4x = 0$ M1

$x = \frac{\arctan(4)}{4}, \frac{\arctan(4)+\pi}{4}, \frac{\arctan(4)+2\pi}{4}, \ldots$

maxima occur at

$x = \frac{\arctan(4)}{4}, \frac{\arctan(4)+2\pi}{4}, \frac{\arctan(4)+4\pi}{4}$ A1

so $y_1 = \mathrm{e}^{-\frac{1}{4}(\arctan(4))}\sin(\arctan(4))$ $(= 0.696)$ A1

$y_2 = \mathrm{e}^{-\frac{1}{4}(\arctan(4)+2\pi)}\sin(\arctan(4) + 2\pi)$ A1

$\left(= \mathrm{e}^{-\frac{1}{4}(\arctan(4)+2\pi)}\sin(\arctan(4)) = 0.145\right)$

$y_3 = \mathrm{e}^{-\frac{1}{4}(\arctan(4)+4\pi)}\sin(\arctan(4) + 4\pi)$ A1

$\left(= \mathrm{e}^{-\frac{1}{4}(\arctan(4)+4\pi)}\sin(\arctan(4)) = 0.0301\right)$ N3

(ii) for finding and comparing $\frac{y_3}{y_2}$ and $\frac{y_2}{y_1}$ M1

$r = \mathrm{e}^{-\frac{\pi}{2}}$ A1

Note: Exact values must be used to gain the M1 and the A1.

[22]

39. **EITHER**

changing to modulus-argument form
$r = 2$

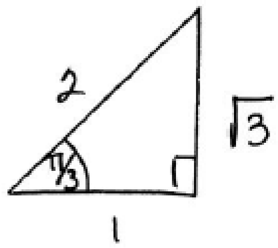
$\theta = \arctan\sqrt{3} = \frac{\pi}{3}$ (M1)A1

$\Rightarrow 1+\sqrt{3}^{\,n} = 2^n\left(\cos\frac{n\pi}{3} + \text{i}\sin\frac{n\pi}{3}\right)$ M1

if $\sin\frac{n\pi}{3} = 0 \Rightarrow n = \{0, \pm 3, \pm 6, ...\}$ (M1)A1 N2

OR

$\theta = \arctan\sqrt{3} = \frac{\pi}{3}$ (M1)(A1)



M1

$n \in \mathbb{R} \Rightarrow \frac{n\pi}{3} = k\pi, k \in \mathbb{Z}$ M1

$\Rightarrow n = 3k, k \in \mathbb{Z}$ A1 N2

[5]

40. (a) $S_6 = 81 \Rightarrow 81 = \frac{6}{2}(2a+5d)$ M1A1

$\Rightarrow 27 = 2a + 5d$

$S_{11} = 231 \Rightarrow 231 = \frac{11}{2}(2a+10d)$ M1A1

$\Rightarrow 21 = a + 5d$

solving simultaneously, $a = 6$, $d = 3$ A1A1

(b) $a + ar = 1$ A1
$a + ar + ar^2 + ar^3 = 5$ A1
$\Rightarrow (a + ar) + ar^2(1 + r) = 5$
$\Rightarrow 1 + ar^2 \times \frac{1}{a} = 5$
obtaining $r^2 - 4 = 0$ M1
$\Rightarrow r = \pm 2$
$r = 2$ (since all terms are positive) A1
$a = \frac{1}{3}$ A1

(c) AP r^{th} term is $3r + 3$ A1
GP r^{th} term is $\frac{1}{3}2^{r-1}$ A1
$3(r + 1) \times \frac{1}{3}2^{r-1} = (r + 1)2^{r-1}$ M1AG

(d) prove: $P_n : \sum_{r=1}^{n}(r+1)2^{r-1} = n2^n, n \in \mathbb{Z}^+$
show true for $n = 1$, i.e.
LHS $= 2 \times 2^0 = 2 =$ RHS A1
assume true for $n = k$, i.e. M1
$\sum_{r=1}^{k}(r+1)2^{r-1} = k2^k, k \in \mathbb{Z}^+$
consider $n = k + 1$
$\sum_{r=1}^{k+1}(r+1)2^{r-1} = k2^k + (k+1)2^k$ M1A1
$= 2^k(k + k + 2)$
$= 2(k + 1)2^k$ A1
$= (k + 1)2^{k+1}$ A1
hence true for $n = k + 1$
P_{k+1} is true whenever P_k is true, and P_1 is true, therefore P_n is true for $n \in \mathbb{Z}^+$ R1

[21]

41. (a) (i) $\frac{1}{z} = \frac{1}{x+\text{i}y} \times \frac{x-\text{i}y}{x-\text{i}y} = \frac{x}{x^2+y^2} - \text{i}\frac{y}{x^2+y^2}$ (M1)A1

(ii) $z + \dfrac{1}{z} = x + \dfrac{x}{x^2+y^2} + \mathrm{i}\left(y - \dfrac{y}{x^2+y^2}\right) = k$ (A1)

for k to be real, $y - \dfrac{y}{x^2+y^2} = 0 \Rightarrow y(x^2 + y^2 - 1) = 0$ M1A1

hence, $y = 0$ or $x^2 + y^2 - 1 = 0 \Rightarrow x^2 + y^2 = 1$ AG

(iii) when $x^2 + y^2 = 1$, $z + \dfrac{1}{z} = 2x$ (M1)A1

$|x| \le 1$ R1

$\Rightarrow |k| \le 2$ AG

(b) (i) $w^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta) = \cos n\theta - \mathrm{i}\sin n\theta$ M1A1

$\Rightarrow w^n + w^{-n} = (\cos n\theta + \mathrm{i}\sin n\theta) + (\cos n\theta - \mathrm{i}\sin n\theta) = 2\cos n\theta$ M1AG

(ii) (rearranging)

$3(w^2 + w^{-2}) - (w + w^{-1}) + 2 = 0$ (M1)

$\Rightarrow 3(2\cos 2\theta) - 2\cos\theta + 2 = 0$ A1

$\Rightarrow 2(3\cos 2\theta - \cos\theta + 1) = 0$

$\Rightarrow 3(2\cos^2\theta - 1) - \cos\theta + 1 = 0$ M1

$\Rightarrow 6\cos^2\theta - \cos\theta - 2 = 0$ A1

$\Rightarrow (3\cos\theta - 2)(2\cos\theta + 1) = 0$ M1

$\therefore \cos\theta = \dfrac{2}{3}, \cos\theta = -\dfrac{1}{2}$ A1A1

$\cos\theta = \dfrac{2}{3} \Rightarrow \sin\theta = \pm\dfrac{\sqrt{5}}{3}$ A1

$\cos\theta = -\dfrac{1}{2} \Rightarrow \sin\theta = \pm\dfrac{\sqrt{3}}{2}$ A1

$\therefore w = \dfrac{2}{3} \pm \dfrac{\mathrm{i}\sqrt{5}}{3}, -\dfrac{1}{2} \pm \dfrac{\mathrm{i}\sqrt{3}}{2}$ A1A1

Note: Allow FT from incorrect $\cos\theta$ and/or $\sin\theta$.

[22]

42. (a) coefficient of x^3 is $\binom{n}{3}\left(\frac{1}{2}\right)^3 = 70$ M1(A1)

$\frac{n!}{3!(n-3)!} \times \frac{1}{8} = 70$ (A1)

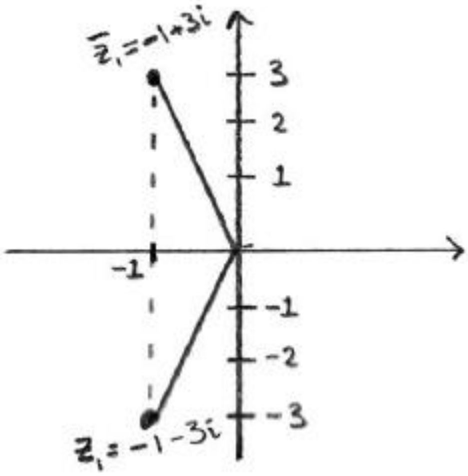
$\Rightarrow \frac{n(n-1)(n-2)}{48} = 70$ (M1)

$n = 16$ A1

(b) $\binom{16}{2}\left(\frac{1}{2}\right)^2 = 30$ A1

[6]

43. (a) one root is $-1 - 3\text{i}$ A1



distance between roots is 6, implies height is 3 (M1)A1

EITHER

$-1 + 3 = 2 \Rightarrow$ third root is 2 A1

OR

$-1 - 3 = -4 \Rightarrow$ third root is -4 A1

(b) **EITHER**

$(z - (-1 + 3i))(z - (-1 - 3i))\ (z - 2) = 0$ M1
$\Rightarrow\ (z^2 + 2z + 10)(z - 2) = 0$ (A1)
$z^3 + 6z - 20 = 0$ A1
$a = 0$, $b = 6$ and $c = -20$

OR

$(z - (-1 + 3i))(z - (-1 - 3i))\ (z + 4) = 0$ M1
$\Rightarrow\ (z^2 + 2z + 10)(z + 4) = 0$ (A1)
$z^3 + 6z^2 + 18z + 40 = 0$ A1
$a = 6$, $b = 18$ and $c = 40$

[7]

44. **METHOD 1**

$r = 2, \theta = -\frac{\pi}{3}$ (A1)(A1)

$\therefore \left(1 - \mathrm{i}\sqrt{3}\right)^{-3} = 2^{-3}\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)^{-3}$ M1

$= \frac{1}{8}\left(\cos\pi + \mathrm{i}\sin\pi\right)$ (M1)

$= -\frac{1}{8}$ A1

METHOD 2

$(1 - \mathrm{i}\sqrt{3})(1 - \mathrm{i}\sqrt{3}) = 1 - 2\mathrm{i}\sqrt{3} - 3 \; (= -2 - 2\mathrm{i}\sqrt{3})$ (M1)A1

$(-2 - 2\mathrm{i}\sqrt{3})(1 - \mathrm{i}\sqrt{3}) = -8$ (M1)(A1)

$\therefore \frac{1}{\left(1 - \mathrm{i}\sqrt{3}\right)^3} = -\frac{1}{8}$ A1

METHOD 3

Attempt at Binomial expansion M1

$(1 - \mathrm{i}\sqrt{3})^3 = 1 + 3(-\mathrm{i}\sqrt{3}) + 3(-\mathrm{i}\sqrt{3})^2 + (-\mathrm{i}\sqrt{3})^3$ (A1)

$= 1 - 3\mathrm{i}\sqrt{3} - 9 + 3\mathrm{i}\sqrt{3}$ (A1)

$= -8$ A1

$\therefore \frac{1}{\left(1 - \mathrm{i}\sqrt{3}\right)^3} = -\frac{1}{8}$ M1

[5]

45. **METHOD 1**

If the areas are in arithmetic sequence, then so are the angles. (M1)

$\Rightarrow S_n = \frac{n}{2}(a+l) \Rightarrow \frac{12}{2}(\theta+2\theta)=18\theta$ M1A1

$\Rightarrow 18\theta = 2\pi$ (A1)

$\theta = \frac{\pi}{9}$ (accept 20°) A1

METHOD 2

$a_{12} = 2a_1$ (M1)

$\frac{12}{2}(a_1 + 2a_1) = \pi r^2$ M1A1

$3a_1 = \frac{\pi r^2}{6}$

$\frac{3}{2}r^2\,\theta = \frac{\pi r^2}{6}$ (A1)

$\theta = \frac{2\pi}{18} = \frac{\pi}{9}$ (accept 20°) A1

METHOD 3

Let smallest angle = a, common difference = d

$a + 11d = 2a$ (M1)

$a = 11d$ A1

$S_n = \frac{12}{2}(2a + 11d) = 2\pi$ M1

$6(2a + a) = 2\pi$ (A1)

$18a = 2\pi$

$a = \frac{\pi}{9}$ (accept 20°) A1

[5]

46. (a) $0 < 2^x < 1$ (M1)

$x < 0$ A1 N2

(b) $\dfrac{35}{1-r} = 40$ M1

$\Rightarrow 40 - 40 \times r = 35$
$\Rightarrow -40 \times r = -5$ (A1)

$\Rightarrow r = 2^x = \dfrac{1}{8}$ A1

$\Rightarrow x = \log_2 \dfrac{1}{8}$ $(= -3)$ A1

Note: The substitution $r = 2^x$ may be seen at any stage in the solution.

[6]

47. (a) $f'(x) = (1 + 2x)\, e^{2x}$ A1

$f'(x) = 0$ M1

$\Rightarrow (1 + 2x)e^{2x} = 0 \Rightarrow x = -\dfrac{1}{2}$ A1

$f''(x) = (2^2 x + 2 \times 2^{2-1})e^{2x} = (4x + 4)e^{2x}$ A1

$f''\left(-\dfrac{1}{2}\right) = \dfrac{2}{e}$ A1

$\dfrac{2}{e} > 0 \Rightarrow$ at $x = -\dfrac{1}{2}$, $f(x)$ has a minimum. R1

$P\left(-\dfrac{1}{2}, -\dfrac{1}{2e}\right)$ A1

(b) $f''(x) = 0 \Rightarrow 4x + 4 = 0 \Rightarrow x = -1$ M1A1

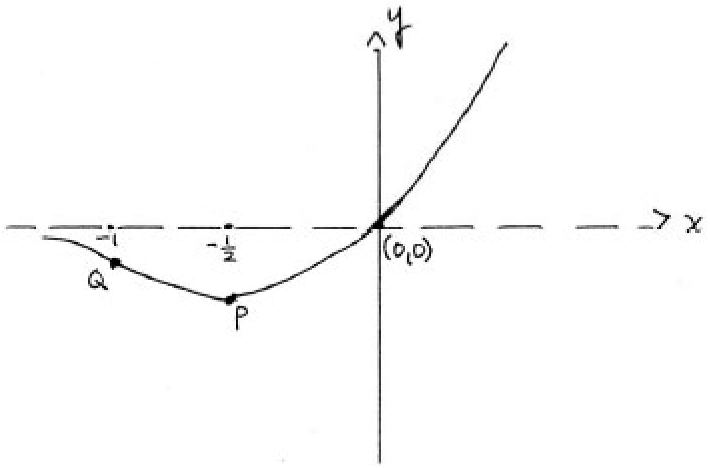
Using the 2nd derivative $f''\left(-\dfrac{1}{2}\right) = \dfrac{2}{e}$ and $f''(-2) = -\dfrac{4}{e^4}$, M1A1

the sign change indicates a point of inflexion. R1

(c) (i) $f(x)$ is concave up for $x > -1$. A1

(ii) $f(x)$ is concave down for $x < -1$. A1

(d)



A1A1A1A1

Note: Award A1 for P and Q, with Q above P,
A1 for asymptote at y = 0,
A1 for (0, 0),
A1 for shape.

(e) Show true for $n = 1$ (M1)

$f'(x) = e^{2x} + 2xe^{2x}$ A1

$= e^{2x}(1 + 2x) = (2x + 2^0)\,e^{2x}$

Assume true for $n = k$, ie $f^{(k)}\,x = (2^k\,x + k \times 2^{k-1})\,e^{2x}$, $k \geq 1$ M1A1

Consider $n = k + 1$, ie an attempt to find $\frac{d}{dx}\left(f^{k}(x)\right)$. M1

$f^{(k+1)}(x) = 2^k\,e^{2x} + 2e^{2x}\,(2^k\,x + k \times 2^{k-1})$ A1

$= (2^k + 2\,(2^k\,x + k \times 2^{k-1}))\,e^{2x}$

$= (2 \times 2^k\,x + 2^k + k \times 2 \times 2^{k-1})\,e^{2x}$

$= (2^{k+1}\,x + 2^k + k \times 2^k)\,e^{2x}$ A1

$= (2^{k+1}\,x + (k + 1)\,2^k)\,e^{2x}$ A1

$P(n)$ is true for $k \Rightarrow P(n)$ is true for $k + 1$, and since true

for $n = 1$, result proved by mathematical induction $\forall\, n \in \mathbb{Z}^+$ R1

Note: Only award R1 if a reasonable attempt is made to prove the $(k + 1)^{th}$ step.

[27]

48. (a) $r = -\frac{1}{3}$ (A1)

$S_\infty = \frac{27}{1+\frac{1}{3}}$ M1

$S_\infty = \frac{81}{4}$ $(=20.25)$ A1 N1

(b) Attempting to show that the result is true for $n = 1$ M1

LHS = a **and** RHS = $\frac{a(1-r)}{1-r} = a$ A1

Hence the result is true for $n = 1$
Assume it is true for $n = k$

$a + ar + ar^2 + ... + ar^{k-1} = \frac{a(1-r^k)}{1-r}$ M1

Consider $n = k + 1$:

$a + ar + ar^2 + ... + ar^{k-1} + ar^k = \frac{a(1-r^k)}{1-r} + ar^k$ M1

$= \frac{a(1-r^k) + ar^k(1-r)}{1-r}$

$= \frac{a - ar^k + ar^k - ar^{k+1}}{1-r}$ A1

Note: Award A1 for an equivalent correct intermediate step.

$= \frac{a - ar^{k+1}}{1-r}$

$= \frac{a(1-r^{k+1})}{1-r}$ A1

Note: Illogical attempted proofs that use the result to be proved would gain M1A0A0 for the last three above marks.

The result is true for $n = k \Rightarrow$ it is true for $n = k + 1$ **and** as it is true for n =1, the result is proved by mathematical induction. R1 N0

Note: To obtain the final R1 mark a reasonable attempt must have been made to prove the $k + 1$ step.

[10]

49. (a) **EITHER**

$w^5 = \left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^5$ (M1)

$= \cos 2\pi + \mathrm{i}\sin 2\pi$ A1

$= 1$ A1

Hence w is a root of $z^5 - 1 = 0$ AG

OR

Solving $z^5 = 1$ (M1)

$z = \cos\frac{2\pi}{5}n + \mathrm{i}\sin\frac{2\pi}{5}n,\quad n = 0, 1, 2, 3, 4.$ A1

$n = 1$ gives $\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$ which is w A1

(b) $(w - 1)(1 + w + w^2 + w^3 + w^4) = w + w^2 + w^3 + w^4 + w^5 - 1$
$- w - w^2 - w^3 - w^4$ M1

$= w^5 - 1$ A1

Since $w^5 - 1 = 0$ **and** $w \neq 1$, $w^4 + w^3 + w^2 + w + 1 = 0$. R1

(c) $1 + w + w^2 + w^3 + w^4 =$

$1+\cos\frac{2\pi}{5}+\mathrm{i}\sin\frac{2\pi}{5}+\left(\cos\frac{2\pi}{5}+\mathrm{i}\sin\frac{2\pi}{5}\right)^2 +$

$\left(\cos\frac{2\pi}{5}+\mathrm{i}\sin\frac{2\pi}{5}\right)^3+\left(\cos\frac{2\pi}{5}+\mathrm{i}\sin\frac{2\pi}{5}\right)^4$ (M1)

$1+\cos\frac{2\pi}{5}+\mathrm{i}\sin\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\mathrm{i}\sin\frac{4\pi}{5}+$

$\cos\frac{6\pi}{5}+\mathrm{i}\sin\frac{6\pi}{5}+\cos\frac{8\pi}{5}+\mathrm{i}\sin\frac{8\pi}{5}$ M1

$1+\cos\frac{2\pi}{5}+\mathrm{i}\sin\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\mathrm{i}\sin\frac{4\pi}{5}+$

$\cos\frac{4\pi}{5}-\mathrm{i}\sin\frac{4\pi}{5}+\cos\frac{2\pi}{5}-\mathrm{i}\sin\frac{2\pi}{5}$ M1A1A1

Notes: Award M1 for attempting to replace 6π and 8π by 4π and 2π
Award A1 for correct cosine terms and A1 for correct sine terms.

$=1+2\cos\frac{4\pi}{5}+2\cos\frac{2\pi}{5}=0$ A1

Note: Correct methods involving equating real parts, use of conjugates or reciprocals are also accepted.

$\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$ AG

Note: Use of cis notation is acceptable throughout this question.

[12]

50. (a) $\left|1+\mathrm{i}\sqrt{3}\right|=2$ or $\left|1-\mathrm{i}\right|=\sqrt{2}$ (A1)

$\arg\left(1+\mathrm{i}\sqrt{3}\right)=\frac{\pi}{3}$ or $\arg(1-\mathrm{i}) = -\frac{\pi}{4}$ $\left(\text{accept } \frac{7\pi}{4}\right)$ (A1)

$|z_1|=2^m$ A1

$|z_2|=\sqrt{2}^n$ A1

$\arg(z_1) = m\arctan\sqrt{3}=m\frac{\pi}{3}$ A1

$\arg(z_2) = n\arctan(-1) = n\frac{-\pi}{4}$ $\left(\text{accept } n\frac{7\pi}{4}\right)$ A1 N2

(b) $2^m = \sqrt{2}^n \Rightarrow n = 2m$ (M1)A1

$m\frac{\pi}{3} = n\frac{-\pi}{4} + 2\pi k$, where k is an integer M1A1

$\Rightarrow m\frac{\pi}{3} + n\frac{\pi}{4} = 2\pi k$

$\Rightarrow m\frac{\pi}{3} + 2m\frac{\pi}{4} = 2\pi k$ (M1)

$\frac{5}{6}m\pi = 2\pi k$

$\Rightarrow m = \frac{12}{5}k$ A1

The smallest value of k such that m is an integer is 5, hence

$m = 12$ A1

$n = 24$. A1 N2

[14]

51. **METHOD 1**

constant term: $\binom{5}{0}(-2x)^0\binom{7}{0}x^0 = 1$ A1

term in x: $\binom{7}{1}x + \binom{5}{1}(-2x) = -3x$ (M1)A1

term in x^2: $\binom{7}{2}x^2 + \binom{5}{2}(-2x)^2 + \binom{7}{1}x\binom{5}{1}(-2x) = -9x^2$ M1A1 N3

METHOD 2

$(1-2x)^5(1+x)^7 = \left(1 + 5(-2x) + \frac{5\times 4(-2x)^2}{2!} + \ldots\right)\left(1 + 7x + \frac{7\times 6}{2}x^2 + \ldots\right)$ M1M1

$= (1 - 10x + 40x^2 + \ldots)(1 + 7x + 21x^2 + \ldots)$

$= 1 + 7x + 21x^2 - 10x - 70x^2 + 40x^2 + \ldots$

$= 1 - 3x - 9x^2 + \ldots$ A1A1A1 N3

[5]

52. $(\sin\theta + \mathrm{i}\,(1 - \cos\theta))^2 = \sin^2\theta - (1 - \cos\theta)^2 + \mathrm{i}\,2\sin\theta(1 - \cos\theta)$ M1A1

Let α be the required argument.

$\tan\alpha = \dfrac{2\sin\theta(1-\cos\theta)}{\sin^2\theta - (1-\cos\theta)^2}$ M1

$= \dfrac{2\sin\theta(1-\cos\theta)}{(1-\cos^2\theta) - (1-2\cos\theta+\cos^2\theta)}$ (M1)

$= \dfrac{2\sin\theta(1-\cos\theta)}{2\cos\theta(1-\cos\theta)}$ A1

$= \tan\theta$ A1

$\alpha = \theta$ A1

[7]

53. **METHOD 1**

Substituting $z = x + \mathrm{i}y$ to obtain $w = \dfrac{x+y\mathrm{i}}{(x+y\mathrm{i})^2+1}$ (A1)

$w = \dfrac{x+y\mathrm{i}}{x^2-y^2+1+2xy\mathrm{i}}$ A1

Use of $(x^2 - y^2 + 1 - 2xy\mathrm{i})$ to make the denominator real. M1

$= \dfrac{(x+y\mathrm{i})(x^2-y^2+1-2xy\mathrm{i})}{(x^2-y^2+1)^2+4x^2y^2}$ A1

$\operatorname{Im} w = \dfrac{y(x^2-y^2+1)-2x^2y}{(x^2-y^2+1)^2+4x^2y^2}$ (A1)

$= \dfrac{y(1-x^2-y^2)}{(x^2-y^2+1)^2+4x^2y^2}$ A1

$\operatorname{Im} w = 0 \Rightarrow 1 - x^2 - y^2 = 0$ i.e. $|z| = 1$ as $y \neq 0$ R1AG N0

METHOD 2

$w(z^2 + 1) = z$ (A1)

$w(x^2 - y^2 + 1 + 2ixy) = x + yi$ A1

Equating real and imaginary parts

$w(x^2 - y^2 + 1) = x$ and $2wx = 1$, $y \neq 0$ M1A1

Substituting $w = \frac{1}{2x}$ to give $\frac{x}{2} - \frac{y^2}{2x} + \frac{1}{2x} = x$ A1

$-\frac{1}{2x}\left(y^2 - 1\right) = \frac{x}{2}$ or equivalent (A1)

$x^2 + y^2 = 1$, i.e. $|z| = 1$ as $y \neq 0$ R1AG

[7]

54. (a) (i) Attempting to find $\boldsymbol{M}^2$ M1

$$\boldsymbol{M}^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$$ A1

$b(a + d) = b$ or $c(a + d) = c$ A1

Hence $a + d = 1$ (as $b \neq 0$ or $c \neq 0$) AG N0

(ii) $a^2 + bc = a$ M1

$\Rightarrow bc = a - a^2$ $(= a(1 - a))$ A1 N1

(b) **METHOD 1**

Using $\det \boldsymbol{M} = ad - bc$ M1

$\det \boldsymbol{M} = ad - a(1 - a)$ or $\det \boldsymbol{M} = a(1 - a) - a(1 - a)$
(or equivalent) A1

$= 0$ using $a + d = 1$ or $d = 1 - a$ to simplify their expression R1

Hence $\boldsymbol{M}$ is a singular matrix AG N0

METHOD 2

Using $bc = a(1 - a)$ and $a + d = 1$ to obtain $bc = ad$ M1A1

$\det \boldsymbol{M} = ad - bc$ and $ad - bc = 0$ as $bc = ad$ R1

Hence $\boldsymbol{M}$ is a singular matrix AG N0

(c) $a(1 - a) > 0$ (M1)

$0 < a < 1$ A1A1 N3

Note: Award A1 for correct endpoints and A1 for correct inequality signs.

(d) **METHOD 1**

Attempting to expand $(\boldsymbol{I} - \boldsymbol{M})^2$ M1

$(\boldsymbol{I} - \boldsymbol{M})^2 = \boldsymbol{I} - 2\boldsymbol{M} + \boldsymbol{M}^2$ A1

$= \boldsymbol{I} - 2\boldsymbol{M} + \boldsymbol{M}$ A1

$= \boldsymbol{I} - \boldsymbol{M}$ AG N0

METHOD 2

Attempting to expand $(\boldsymbol{I} - \boldsymbol{M})^2 = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}^2$ (or equivalent) M1

$(\boldsymbol{I} - \boldsymbol{M})^2 = \begin{pmatrix} (1-a)^2 + bc & -b(1-a) - b(1-d) \\ -c(1-a) - c(1-d) & bc + (1-d)^2 \end{pmatrix}$

(or equivalent) A1

Use of $a + d = 1$ and $bc = a - a^2$ to show desired result. M1

Hence $(\boldsymbol{I} - \boldsymbol{M})^2 = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}$ AG N0

(e) (Let $P(n)$ be $(\boldsymbol{I} - \boldsymbol{M})^n = \boldsymbol{I} - \boldsymbol{M}$)

For $n = 1$: $(\boldsymbol{I} - \boldsymbol{M})^1 = \boldsymbol{I} - \boldsymbol{M}$, so $P(1)$ is true A1

Assume $P(k)$ is true, i.e. $(\boldsymbol{I} - \boldsymbol{M})^k = \boldsymbol{I} - \boldsymbol{M}$ M1

Consider $P(k + 1)$

$(\boldsymbol{I} - \boldsymbol{M})^{k+1} = (\boldsymbol{I} - \boldsymbol{M})^k (\boldsymbol{I} - \boldsymbol{M})$ M1

$= (\boldsymbol{I} - \boldsymbol{M})(\boldsymbol{I} - \boldsymbol{M})$ $(= (\boldsymbol{I} - \boldsymbol{M})^2)$ A1

$= (\boldsymbol{I} - \boldsymbol{M})$ A1

$P(k)$ true implies $P(k + 1)$ true, $P(1)$ true so $P(n)$ true $\forall n \in \mathbb{Z}^+$ R1 N0

[20]

55. (a) $P(Z = n) = \sum_{k=0}^{n} e^{-\lambda} \times \frac{\lambda^k}{k!} \times e^{-\mu} \times \frac{\mu^{n-k}}{(n-k)!}$ M1A1

$= \frac{e^{-(\mu+\lambda)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda^k \mu^{n-k}$ M1A1

$= \frac{e^{-(\mu+\lambda)}}{n!} (\lambda + \mu)^n$ A1

This shows that Z is Poisson distributed with mean $(\lambda + \mu)$. R1

(b) The result is (trivially) true for $n = 1$. A1

Assuming it to be true for $n = k$, i.e. $\sum_{r=1}^{k} U_r \sim \text{Po}(km)$ M1

Consider $\sum_{r=1}^{k+1} U_r = \sum_{r=1}^{k} U_r + U_{k+1}$ M1A1

which, using (a) is Po($km + m$) i.e. Po($[k + 1]m$) A1

Hence proved by induction since true for $n = k \Rightarrow$ true for $n = k + 1$ and we have shown true for $n = 1$. R1

[12]

56. $81 = \frac{n}{2}(1.5 + 7.5)$ M1

$\Rightarrow n = 18$ A1

$1.5 + 17d = 7.5$ M1

$\Rightarrow d = \frac{6}{17}$ A1 N0

[4]

57. (a) Let $n = 1$

LHS = $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

RHS = $\begin{pmatrix} \cos 1\theta & -\sin 1\theta \\ \sin 1\theta & \cos 1\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Hence true for $n = 1$ M1A1

Assume true for $n = k$ M1

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$ A1

Result for $k + 1$ is M1

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{k+1} = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ A1

$= \begin{pmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & -\cos k\theta\sin\theta - \sin k\theta\cos\theta \\ \sin k\theta\cos\theta + \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{pmatrix}$ A1

$= \begin{pmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{pmatrix}$ A1

Hence if true for k, true for $k + 1$. However, result is true for $k = 1$. R1
Hence proved by induction.

(b) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^1 = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ M1A1A1

Note: Award M1 for any recognition that the inverse is required.

A1 for $\dfrac{1}{\cos^2\theta+\sin^2\theta}$ and A1 for $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$.

$= \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ A1

Now $\begin{pmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ A1

Hence true. AG

[14]

58. (a) $z = (1-\mathrm{i})^{\frac{1}{4}}$

Let $1 - \mathrm{i} = r(\cos\theta + \mathrm{i}\sin\theta)$

$\Rightarrow r = \sqrt{2}$ A1

$\theta = -\dfrac{\pi}{4}$ A1

$z = \left(\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{4}\right)\right)\right)^{\frac{1}{4}}$ M1

$= \left(\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}+2n\pi\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{4}+2n\pi\right)\right)\right)^{\frac{1}{4}}$

$= 2^{\frac{1}{8}}\left(\cos\left(-\dfrac{\pi}{16}+\dfrac{n\pi}{2}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{16}+\dfrac{n\pi}{2}\right)\right)$ M1

$= 2^{\frac{1}{8}}\left(\cos\left(-\dfrac{\pi}{16}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{16}\right)\right)$

Note: Award M1 above for this line if the candidate has forgotten to add 2π and no other solution given.

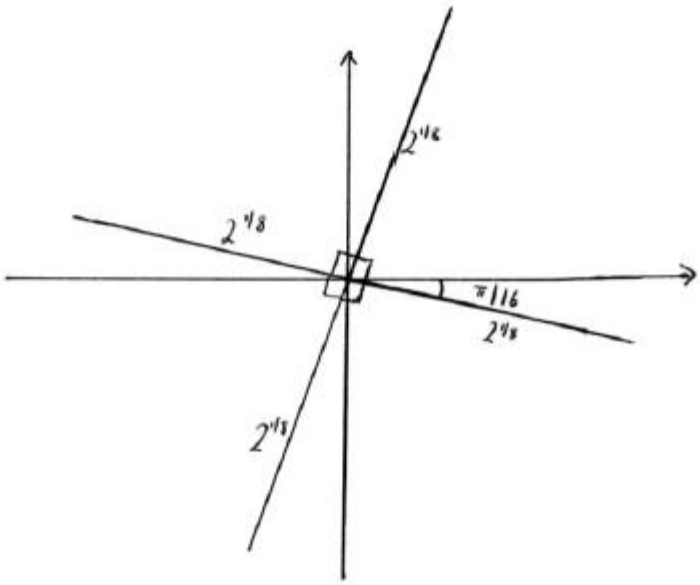
$= 2^{\frac{1}{8}}\left(\cos\left(\dfrac{7\pi}{16}\right)+\mathrm{i}\sin\left(\dfrac{7\pi}{16}\right)\right)$

$= 2^{\frac{1}{8}}\left(\cos\left(\dfrac{15\pi}{16}\right)+\mathrm{i}\sin\left(\dfrac{15\pi}{16}\right)\right)$

$= 2^{\frac{1}{8}}\left(\cos\left(-\dfrac{9\pi}{16}\right)+\mathrm{i}\sin\left(-\dfrac{9\pi}{16}\right)\right)$ A2

Note: Award A1 for 2 correct answers. Accept any equivalent form.

(b)



A2

Note: Award A1 for roots being shown equidistant from the origin and one in each quadrant.
A1 for correct angular positions. It is not necessary to see written evidence of angle, but must agree with the diagram.

(c) $$\frac{z_2}{z_1} = \frac{2^{\frac{1}{8}}\left(\left(\cos\frac{15\pi}{16}\right)+\mathrm{i}\sin\left(\frac{15\pi}{16}\right)\right)}{2^{\frac{1}{8}}\left(\left(\cos\frac{7\pi}{16}\right)+\mathrm{i}\sin\left(\frac{7\pi}{16}\right)\right)}$$ M1A1

$= \cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}$ (A1)

$= \mathrm{i}$ A1 N2

($\Rightarrow$ $a = 0$, $b = 1$)

[12]

59. (a) $(x-1)(x^4+x^3+x^2+x+1)$

$= x^5+x^4+x^3+x^2+x-x^4-x^3-x^2-x-1$ (M1)

$= x^5-1$ A1

(b) b is a root

$f(b) = 0$

$b^5 = 1$ M1

$b^5 - 1 = 0$ A1

$(b-1)(b^4+b^3+b^2+b+1) = 0$

$b \neq 1$ R1

$1+b+b^2+b^3+b^4 = 0$ as shown. AG

(c) (i) $u + v = b^4 + b^3 + b^2 + b = -1$ A1

$uv = (b + b^4)(b^2 + b^3) = b^3 + b^4 + b^6 + b^7$ A1

Now $b^5 = 1$ (A1)

Hence $uv = b^3 + b^4 + b + b^2 = -1$ A1

Hence $u + v = uv = -1$ AG

(ii) $(u - v)^2 = (u^2 + v^2) - 2uv$ (M1)

$= ((u + v)^2 - 2uv) - 2uv \quad (= (u + v)^2 - 4uv)$ (M1)A1

Given $u - v > 0$

$u - v = \sqrt{(u+v)^2 - 4uv}$

$= \sqrt{(-1)^2 - 4(-1)}$

$= \sqrt{1+4}$ A1

$= \sqrt{5}$ AG

Note: Award A0 unless an indicator is given that $u - v = -\sqrt{5}$ is invalid.

[13]

60. $2 \times 1.05^{n-1} > 500$ M1

$n - 1 > \dfrac{\log 250}{\log 1.05}$ M1

$n - 1 > 113.1675...$ A1

$n = 115$ (A1)

$u_{115} = 521$ A1 N5

Note: Accept graphical solution with appropriate sketch.

[5]

61. (a) Boys can be chosen in $\dfrac{6 \times 5}{2} = 15$ ways (A1)

Girls can be chosen in $\dfrac{5 \times 4}{2} = 10$ ways (A1)

Total = 15 × 10 = 150 ways A1

(b) Number of ways = 5 × 4 = 20 (M1)A1

(c) $\dfrac{20}{150}\left(= \dfrac{2}{15}\right)$ A1

(d) **METHOD 1**

$P(T) = \frac{1}{5}$; $P(A) = \frac{2}{5}$ A1

$P(T \text{ or } A \text{ but not both}) = P(T) \times P(A') + P(T') \times P(A)$ M1A1

$= \frac{1}{5}\times\frac{3}{5}+\frac{4}{5}\times\frac{2}{5}=\frac{11}{25}$ A1

METHOD 2

Number of selections including Fred = $5 \times \binom{5}{2} = 50$ A1

Number of selections including Tim but not Anna = $\binom{4}{2} = 6$ A1

Number of selections including Anna but not Tim = 4 × 4 = 16

Note: Both statements are needed to award A1.

$P(T \text{ or } A \text{ but not both}) = \frac{6+16}{50}=\frac{11}{25}$ M1A1

[10]

62. **EITHER**

$4 \ln 2 – 3 \ln 2^2 = –\ln k$ M1
$4 \ln 2 – 6 \ln 2 = –\ln k$ (M1)
$–2 \ln 2 = –\ln k$ (A1)
$–\ln 2^2 = –\ln k$ M1
$k = 4$ A1

OR

$\ln 2^4 – \ln 4^3 = –\ln k$ M1

$\ln \frac{2^4}{4^3} = \ln k^{-1}$ M1A1

$\frac{2^4}{4^3}=\frac{1}{k}$ A1

$\Rightarrow k = \frac{4^3}{2^4}=\frac{64}{16} = 4$ A1

[5]

63. $\log_3(x + 17) - 2 = \log_3 2x$

$\log_3(x + 17) - \log_3 2x = 2$

$\log_3\left(\frac{x+17}{2x}\right) = 2$ M1A1

$\frac{x+17}{2x} = 9$ M1A1

$x + 17 = 18x$

$17 = 17x$

$x = 1$ A1

[5]

64. $2^{2x+2} - 10 \times 2^x + 4 = 0$

$y = 2^x$

$4y^2 - 10y + 4 = 0$ M1A1

$2y^2 - 5y + 2 = 0$

By factorisation or using the quadratic formula (M1)

$y = \frac{1}{2}$ $y = 2$ A1

$2^x = \frac{1}{2}$ $2^x = 2$

$x = -1$ $x = 1$ A1A1

[6]

65. $a^2 + 2\text{i}ab - b^2 = 3 + 4\text{i}$

Equate real and imaginary parts (M1)

$a^2 - b^2 = 3$, $2ab = 4$ A1

Since $b = \frac{2}{a}$

$\Rightarrow a^2 - \frac{4}{a^2} = 3$ (M1)

$\Rightarrow a^4 - 3a^2 - 4 = 0$ A1

Using factorisation or the quadratic formula (M1)

$\Rightarrow a = \pm 2$

$\Rightarrow b = \pm 1$

$\Rightarrow \sqrt{3+4\text{i}} = 2 + \text{i}, -2 - \text{i}$ A1A1

[7]

66. 2 + i is a root $\Rightarrow$ 2 – i is a root R1

$[x-(2+\mathrm{i})]\,[x-(2-\mathrm{i})]$ are factors M1
$= x^2-(2-\mathrm{i})\,x-(2+\mathrm{i})\,x+(2+\mathrm{i})(2-\mathrm{i})$
$= x^2-2\,x+\mathrm{i}x-2\,x-\mathrm{i}x+(4+1)$ (A1)
$= x^2-4x+5$ A1
Hence $x-2$ is a factor $\Rightarrow$ 2 is a root R1

[5]

67. $5zz^*+10=(6-18\mathrm{i})z^*$ M1
Let $z=a+\mathrm{i}b$
$5\times10+10=(6-18\mathrm{i})(a-b\mathrm{i})$ $(=6a-6b\mathrm{i}-18a\mathrm{i}-18b)$ M1A1
Equate real and imaginary parts (M1)
$\Rightarrow$ $6a-18b=60$ and $6b+18a=0$
$\Rightarrow$ $a=1$ and $b=-3$ A1A1
$z=1-3\mathrm{i}$ A1

[7]

68. $8\mathrm{i}=8\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2n\pi\right)}$ (M1)

For $n=0$

$(8\mathrm{i})^{\frac{1}{3}}=2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ (M1)

$=2\cos\frac{\pi}{6}+2\mathrm{i}\sin\frac{\pi}{6}$ A1

$=\sqrt{3}+\mathrm{i}$ A1

For $n=1$

$(8\mathrm{i})^{\frac{1}{3}}=2\cos\frac{5\pi}{6}+2\mathrm{i}\sin\frac{5\pi}{6}$ M1

$=-\sqrt{3}+\mathrm{i}$ A1

For $n=2$

$(8\mathrm{i})^{\frac{1}{3}}=2\cos\frac{3\pi}{2}+2\mathrm{i}\sin\frac{3\pi}{2}$ M1

$=-2\mathrm{i}$ A1

[8]

69. $\mathrm{i}z_1 + 2z_2 = 3 \Rightarrow z_2 = -\frac{1}{2}\mathrm{i}z_1 + \frac{3}{2}$

$z_1 + (1 - \mathrm{i})z_2 = 4$

$\Rightarrow z_1 + (1 - \mathrm{i})\left(-\frac{1}{2}\mathrm{i}z_1 + \frac{3}{2}\right) = 4$ M1A1

$\Rightarrow z_1 - \frac{1}{2}\mathrm{i}z_1 + \frac{3}{2} + \frac{1}{2}\mathrm{i}^2 z_1 - \frac{3}{2}\mathrm{i} = 4$

$\Rightarrow \frac{1}{2}z_1 - \frac{1}{2}\mathrm{i}z_1 = \frac{5}{2} + \frac{3}{2}\mathrm{i}$

$\Rightarrow z_1 - \mathrm{i}z_1 = 5 + 3\mathrm{i}$ A1

EITHER

Let $z_1 = x + \mathrm{i}y$ (M1)

$\Rightarrow x + \mathrm{i}y - \mathrm{i}x - \mathrm{i}^2 y = 5 + 3\mathrm{i}$

Equate real and imaginary parts M1

$\Rightarrow x + y = 5$

$\underline{-x + y = 3}$

$2y = 8$

$y = 4 \Rightarrow x = 1$ i.e. $z_1 = 1 + 4\mathrm{i}$ A1A1

$z_2 = -\frac{1}{2}\mathrm{i}(1 + 4\mathrm{i}) + \frac{3}{2}$ M1

$z_2 = -\frac{1}{2}\mathrm{i} - 2\mathrm{i}^2 + \frac{3}{2}$

$z_2 = \frac{7}{2} - \frac{1}{2}\mathrm{i}$ A1

OR

$z_1 = \frac{5 + 3\mathrm{i}}{1 - \mathrm{i}}$ M1

$z_1 = \frac{(5 + 3\mathrm{i})(1 + \mathrm{i})}{(1 - \mathrm{i})(1 + \mathrm{i})} \left(= \frac{5 + 8\mathrm{i} - 3}{2}\right)$ M1A1

$z_1 = 1 + 4\mathrm{i}$ A1

$z_2 = -\frac{1}{2}\mathrm{i}(1 + 4\mathrm{i}) + \frac{3}{2}$ M1

$z_2 = -\frac{1}{2}\mathrm{i} - 2\mathrm{i}^2 + \frac{3}{2}$

$z_2 = \frac{7}{2} - \frac{1}{2}\mathrm{i}$ A1

[9]

70. **METHOD 1**

$20 + 10bi = (1 - bi)(-7 + 9i)$ (M1)
$20 + 10bi = (-7 + 9b) + (9 + 7b)i$ A1A1
Equate real and imaginary parts (M1)

EITHER

$-7 + 9b = 20$
$b = 3$ (M1)A1

OR

$10b = 9 + 7b$
$3b = 9$
$b = 3$ (M1)A1

METHOD 2

$= \frac{(2+bi)(1+bi)}{(1-bi)(1+bi)} = \frac{-7+9i}{10}$ (M1)

$\frac{2-b^2+3bi}{1+b^2} = \frac{-7+9i}{10}$ A1

Equate real and imaginary parts (M1)

$\frac{2-b^2}{1+b^2} = -\frac{7}{10}$ Equation A

$\frac{3b}{1+b^2} = \frac{9}{10}$ Equation B

From equation A
$20 - 10b^2 = -7 - 7b^2$
$3b^2 = 27$
$b = \pm 3$ A1

From equation B
$30b = 9 + 9b^2$
$3b^2 - 10b + 3 = 0$
By factorisation or using the quadratic formula
$b = \frac{1}{3}$ or 3 A1
Since 3 is the common solution to both equations $b = 3$ R1

[6]

71. Let $f(n) = 5^n + 9^n + 2$ and let P_n be the proposition that $f(n)$ is divisible by 4.

Then $f(1) = 16$ A1

So P_1 is true A1

Let P_n be true for $n = k$, i.e. $f(k)$ is divisible by 4 M1

Consider $f(k+1) = 5^{k+1} + 9^{k+1} + 2$ M1

$= 5^k(4+1) + 9^k(8+1) + 2$ A1

$= f(k) + 4(5^k + 2 \times 9^k)$ A1

Both terms are divisible by 4 so $f(k+1)$ is divisible by 4. R1

P_k true $\Rightarrow$ P_{k+1} true R1

Since P_1 is true, P_n is proved true by mathematical induction for $n \in \mathbb{Z}^+$. R1 N0

[9]

72. **METHOD 1**

since $b > 0$ (M1)

$\Rightarrow$ $\arg(b + \text{i}) = 30°$ A1

$\frac{1}{b} = \tan 30°$ M1A1

$b = \sqrt{3}$ A2 N2

METHOD 2

$\arg(b + \text{i})^2 = 60°$ $\Rightarrow$ $\arg(b^2 - 1 + 2b\text{i}) = 60°$ M1

$\frac{2b}{(b^2 - 1)} = \tan 60° = \sqrt{3}$ M1A1

$\sqrt{3}b^2 - 2b - \sqrt{3} = 0$ A1

$(\sqrt{3}b + 1)(b - \sqrt{3}) = 0$

since $b > 0$ (M1)

$b = \sqrt{3}$ A1 N2

[6]

73. Probability $= \frac{{}^3C_2 \times {}^6C_1}{{}^9C_3}$ M1A1A1A1

$= \frac{3 \times 6 \times 3! \times 6!}{9!} = \frac{3 \times 6 \times 6}{9 \times 8 \times 7} = \frac{3}{14}$ A1

[5]

74. (a) Let $p = 2$, $\Rightarrow$ $8 + 4 - 10 - 2 = 0$ M1

Since this fits $p = 2$ is a solution. R1

(b) $p^3 + p^2 - 5p - 2 = (p - 2)(p^2 + ap + b)$

$= p^3 + ap^2 + bp - 2p^2 - 2ap - 2b$ M1A1

$= p^3 + p^2(a - 2) + p(b - 2a) - 2b$

Equate constants $\Rightarrow$ $-2 = -2b$

$b = 1$ A1

Equate coefficients of p^2 $\Rightarrow$ $a - 2 = 1$

$a = 3$ A1

(c) $p^2 + 3p + 1 = 0$ M1

$p = \dfrac{-3 \pm \sqrt{9-4}}{2} = \dfrac{-3 \pm \sqrt{5}}{2}$ A1A1

(d) (i) Arithmetic sequence: $1, 1 + p, 1 + 2p, 1 + 3p$ A1

Geometric sequence: $1, p, p^2, p^3$ A1

(ii) $(1 + 2p) + (1 + 3p) = p^2 + p^3$ M1A1

$\Rightarrow$ $p^3 + p^2 - 5p - 2 = 0$ A1

Therefore, from part (i), $p = 2$, $p = \dfrac{-3 \pm \sqrt{5}}{2}$ R1

(iii) The sum to infinity of a geometric series exists if $|p| < 1$. R1

Hence, $p = \dfrac{-3 + \sqrt{5}}{2}$ is the only such number. A1

(iv) The sum of the first 20 terms of the arithmetic series can be found by applying the sum formula

$S_{20} = 10(2a + 19d) = 10(2 + 19p)$ M1A1

So, $S_{20} = 10\left(2 + 19\left(\dfrac{\sqrt{5} - 3}{2}\right)\right) = -265 + 95\sqrt{5}$ A1A1A1

[22]

75. (a) $z = \dfrac{\frac{1}{2}e^{2i\theta}}{e^{i\theta}}$ (M1)

$z = \frac{1}{2}e^{i\theta}$ A1 N2

(b) $|z| = \frac{1}{2}$ A2

$|z| < 1$ AG

(c) Using $S_\infty = \dfrac{a}{1-r}$ (M1)

$S_\infty = \dfrac{e^{i\theta}}{1-\frac{1}{2}e^{i\theta}}$ A1 N2

(d) (i) $S_\infty = \dfrac{e^{i\theta}}{1-\frac{1}{2}e^{i\theta}} = \dfrac{\text{cis}\,\theta}{1-\frac{1}{2}\text{cis}\,\theta}$ (M1)

$\dfrac{\cos\theta + i\sin\theta}{1-\frac{1}{2}(\cos\theta + i\sin\theta)}$ (A1)

Also $S_\infty = e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots$

$= \text{cis}\,\theta + \frac{1}{2}\text{cis}2\theta + \frac{1}{4}\text{cis}3\theta + \ldots$ (M1)

$S_\infty = \left(\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots\right) + i\left(\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{4}\sin 3\theta + \ldots\right)$ A1

(ii) Taking real parts,

$$\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{4}\cos 3\theta+\ldots=\mathrm{Re}\left(\frac{\cos\theta+\mathrm{i}\sin\theta}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)}\right)$$ A1

$$=\ \mathrm{Re}\left(\frac{(\cos\theta+\mathrm{i}\sin\theta)}{\left(1-\frac{1}{2}\cos\theta-\frac{1}{2}\mathrm{i}\sin\theta\right)}\times\frac{1-\frac{1}{2}\cos\theta+\frac{1}{2}\mathrm{i}\sin\theta}{\left(1-\frac{1}{2}\cos\theta+\frac{1}{2}\mathrm{i}\sin\theta\right)}\right)$$ M1

$$=\frac{\cos\theta-\frac{1}{2}\cos^2\theta-\frac{1}{2}\sin^2\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\frac{1}{4}\sin^2\theta}$$ A1

$$=\frac{\left(\cos\theta-\frac{1}{2}\right)}{1-\cos\theta+\frac{1}{4}(\sin^2\theta+\cos^2\theta)}$$ A1

$$=\frac{(2\cos\theta-1)\div 2}{(4-4\cos\theta+1)\div 4}=\frac{4(2\cos\theta-1)}{2(5-4\cos\theta)}$$ A1

$$=\frac{4\cos\theta-2}{5-4\cos\theta}$$ A1AG N0

[25]

76. (a) $z=\dfrac{-2\pm\sqrt{4-16}}{2}=-1\pm\mathrm{i}\sqrt{3}$ M1

$-1+\ \mathrm{i}\sqrt{3}=r\mathrm{e}^{\mathrm{i}\theta}\Rightarrow\ r=2$ A1

$\theta=\arctan\ \dfrac{\sqrt{3}}{-1}=\dfrac{2\pi}{3}$ A1

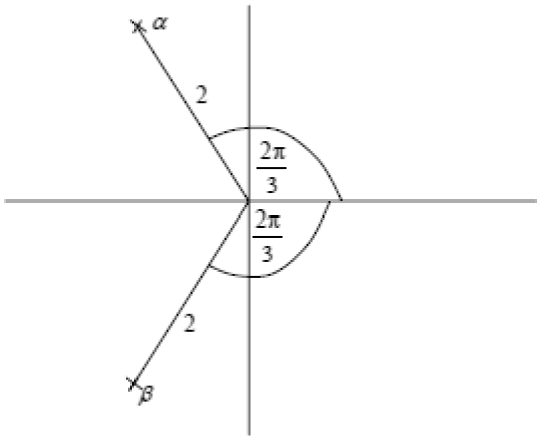
$-1-\ \mathrm{i}\sqrt{3}\ =r\mathrm{e}^{\mathrm{i}\theta}\ \Rightarrow\ r=2$

$\theta=\arctan\ \dfrac{\sqrt{3}}{-1}=-\dfrac{2\pi}{3}$ A1

$\Rightarrow\alpha=2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ A1

$\Rightarrow\beta=2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$ A1

(b)



A1A1

(c) $\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n$

Let $n = 1$

Left hand side $= \cos 1\theta + \mathrm{i}\sin 1\theta = \cos\theta + \mathrm{i}\sin\theta$

Right hand side $= (\cos\theta + \mathrm{i}\sin\theta)^1 = \cos\theta + \mathrm{i}\sin\theta$

Hence true for $n = 1$ M1A1

Assume true for $n = k$ M1

$\cos k\theta + \mathrm{i}\sin k\theta = (\cos\theta + \mathrm{i}\sin\theta)^k$

$\Rightarrow \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta = (\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)$ M1A1

$= (\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta)$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + \mathrm{i}(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$ A1

$= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta$ A1

Hence if true for $n = k$, true for $n = k + 1$

However if it is true for $n = 1$

$\Rightarrow$ true for $n = 2$ *etc.* R1

$\Rightarrow$ hence proved by induction

(d) $\dfrac{\alpha^3}{\beta^2} = \dfrac{8\mathrm{e}^{\mathrm{i}2\pi}}{4\mathrm{e}^{-\mathrm{i}\frac{4\pi}{3}}} = 2\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}$ A1

$= 2\cos\dfrac{4\pi}{3} + 2\mathrm{i}\sin\dfrac{4\pi}{3}$ (M1)

$= -\dfrac{2}{2} - 2\dfrac{\mathrm{i}\sqrt{3}}{2} = -1 - \mathrm{i}\sqrt{3}$ A1A1

(e) $\alpha^3 = 8\mathrm{e}^{\mathrm{i}2\pi}$ A1

$\beta^3 = 8\mathrm{e}^{-\mathrm{i}2\pi}$ A1

Since $\mathrm{e}^{2\pi}$ and $\mathrm{e}^{-2\pi}$ are the same $\alpha^3 = \beta^3$ R1

(f) **EITHER**

$\alpha = -1 + \mathrm{i}\sqrt{3}$ $\quad\beta = -1 - \mathrm{i}\sqrt{3}$

$\alpha^* = -1 - \mathrm{i}\sqrt{3}$ $\quad\beta^* = -1 + \mathrm{i}\sqrt{3}$ A1

$\alpha\beta^* = (-1 + \mathrm{i}\sqrt{3})(-1 + \mathrm{i}\sqrt{3}) = 1 - 2\mathrm{i}\sqrt{3} - 3 = 2 - 2\mathrm{i}\sqrt{3}$ M1A1

$\beta\alpha^* = (-1 - \mathrm{i}\sqrt{3})(-1 - \mathrm{i}\sqrt{3}) = 1 + 2\mathrm{i}\sqrt{3} - 3 = -2 + 2\mathrm{i}\sqrt{3}$ A1

$\Rightarrow \alpha\beta^* + \beta\alpha^* = -4$ A1

OR

Since $\alpha^* = \beta$ and $\beta^* = \alpha$

$\alpha\beta^* = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} \times 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} = 4\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}$ M1A1

$\beta\alpha^* = 2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}} \times 2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}} = 4\mathrm{e}^{-\mathrm{i}\frac{4\pi}{3}}$ A1

$\alpha\beta^* + \beta\alpha^* = 4\left(\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}} + \mathrm{e}^{-\mathrm{i}\frac{4\pi}{3}}\right)$

$= 4\left(\cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3} + \cos\frac{4\pi}{3} - \mathrm{i}\sin\frac{4\pi}{3}\right)$ A1

$= 8\cos\frac{4\pi}{3} = 8 \times -\frac{1}{2} = -4$ A1

(g) $\alpha^n = 2^n \mathrm{e}^{\mathrm{i}2\frac{\pi n}{3}}$ M1A1

This is real when n is a multiple of 3 R1

i.e. $n = 3N$ where $N \in \mathbb{Z}^+$

[31]

77. (a) $5000(1.063)^n$ A1 N1

(b) Value = \$ $5000(1.063)^5$ (= \$ 6786.3511…)

= \$ 6790 to 3 s.f. (accept \$ 6786, or \$ 6786.35) A1 N1

(c) (i) $5000(1.063)^n > 10\,000$ (or $(1.063)^n > 2$) A1 N1

(ii) Attempting to solve the above inequality $n \log(1.063) > \log 2$ (M1)
$n > 11.345...$ (A1)
12 years A1 N3

Note: Candidates are likely to use TABLE or LIST on a GDC to find *n*. A good way of communicating this is suggested below.

Let $y = 1.063^x$ (M1)
When $x = 11$, $y = 1.9582$, when $x = 12$, $y = 2.0816$ (A1)
$x = 12$ i.e. 12 years A1 N3

[6]