MATH HL2 EXAM PREP – CORE TOPICS - ALGEBRA

1. Given that $\frac{z}{z+2} = 2 - \mathrm{i}$, $z \in \mathbb{C}$, find z in the form $a + \mathrm{i}b$.

(Total 4 marks)

2. A geometric sequence $u_1, u_2, u_3, \ldots$ has $u_1 = 27$ and a sum to infinity of $\frac{81}{2}$.

(a) Find the common ratio of the geometric sequence.

(2)

An arithmetic sequence $v_1, v_2, v_3, \ldots$ is such that $v_2 = u_2$ and $v_4 = u_4$.

(b) Find the greatest value of N such that $\sum_{n=1}^{N} v_n > 0$.

(5)
(Total 7 marks)

3. Prove by mathematical induction that, for $n \in \mathbb{Z}^+$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

(Total 8 marks)

4. Two players, A and B, alternately throw a fair six–sided dice, with A starting, until one of them obtains a six. Find the probability that A obtains the first six.

(Total 7 marks)

5. (a) Show that $\sin 2nx = \sin((2n + 1)x)\cos x - \cos((2n + 1)x)\sin x$.

(2)

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2\sin x},$$

for all $n \in \mathbb{Z}^+$, $\sin x \neq 0$.

(12)

(c) Solve the equation $\cos x + \cos 3x = \frac{1}{2}$, $0 < x < \pi$.

(6)
(Total 20 marks)

6. (a) Consider the following sequence of equations.

$$1 \times 2 = \frac{1}{3}(1 \times 2 \times 3),$$
$$1 \times 2 + 2 \times 3 = \frac{1}{3}(2 \times 3 \times 4),$$
$$1 \times 2 + 2 \times 3 + 3 \times 4 = \frac{1}{3}(3 \times 4 \times 5),$$
$$\ldots .$$

(i) Formulate a conjecture for the n^{th} equation in the sequence.

(ii) Verify your conjecture for $n = 4$.

(2)

(b) A sequence of numbers has the n^{th} term given by $u_n = 2^n + 3$, $n \in \mathbb{Z}^+$. Bill conjectures that all members of the sequence are prime numbers. Show that Bill's conjecture is false.

(2)

(c) Use mathematical induction to prove that $5 \times 7^n + 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

(6)
(Total 10 marks)

7. (a) Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^3$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$.

(2)

(b) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(3)

(c) Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(3)

(d) **Hence** solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(6)

(e) By considering the solutions of the equation $\cos 5\theta = 0$, show that

$\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$.

(8)
(Total 22 marks)

8. The complex numbers $z_1 = 2 - 2\mathrm{i}$ and $z_2 = 1 - \mathrm{i}\sqrt{3}$ are represented by the points A and B respectively on an Argand diagram. Given that O is the origin,

(a) find AB, giving your answer in the form $a\sqrt{b - \sqrt{3}}$, where $a, b \in \mathbb{Z}^+$;

(3)

(b) calculate $\mathrm{A\hat{O}B}$ in terms of π.

(3)
(Total 6 marks)

• $**9.** In the arithmetic series with n^{th} term u_n, it is given that $u_4 = 7$ and $u_9 = 22$.
Find the minimum value of n so that $u_1 + u_2 + u_3 + ... + u_n > 10\,000$.

(Total 5 marks)

10. An arithmetic sequence has first term a and common difference d, $d \neq 0$.
The 3rd, 4th and 7th terms of the arithmetic sequence are the first three terms of a geometric sequence.

(a) Show that $a = -\frac{3}{2}d$.

(3)

(b) Show that the 4th term of the geometric sequence is the 16th term of the arithmetic sequence.

(5)
(Total 8 marks)

11. (a) Given that $\boldsymbol{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, show that $\boldsymbol{A}^2 = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$.

(3)

(b) Prove by induction that

$$\boldsymbol{A}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}, \text{ for all } n \in \mathbb{Z}^+.$$

(7)

(c) Given that $\boldsymbol{A}^{-1}$ is the inverse of matrix $\boldsymbol{A}$, show that the result in part (b) is true where $n = -1$.

(3)
(Total 13 marks)

12. Consider $\omega = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$.

(5)

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \frac{\pi}{2}$ on an Argand diagram.

(4)

(c) (i) Expand and simplify $F(z) = (z - 1)(z - \omega)(z - \omega^2)$ where z is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω.

(7)
(Total 16 marks)

13. (a) Factorize $z^3 + 1$ into a linear and quadratic factor.

(2)

Let $\gamma = \dfrac{1+\mathrm{i}\sqrt{3}}{2}$.

(b) (i) Show that γ is one of the cube roots of -1.

(ii) Show that $\gamma^2 = \gamma - 1$.

(iii) Hence find the value of $(1 - \gamma)^6$.

(9)

The matrix $\boldsymbol{A}$ is defined by $\boldsymbol{A} = \begin{pmatrix} \gamma & 1 \\ 0 & \dfrac{1}{\gamma} \end{pmatrix}$.

(c) Show that $\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I} = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix.

(4)

(d) Deduce that

(i) $\boldsymbol{A}^3 = -\boldsymbol{I}$;

(ii) $\boldsymbol{A}^{-1} = \boldsymbol{I} - \boldsymbol{A}$.

(5)
(Total 20 marks)

14. (a) Write down the quadratic expression $2x^2 + x - 3$ as the product of two linear factors.

(1)

(b) Hence, or otherwise, find the coefficient of x in the expansion of $(2x^2 + x - 3)^8$.

(4)
(Total 5 marks)

15. Solve the following system of equations.

$$\log_{x+1} y = 2$$
$$\log_{y+1} x = \frac{1}{4}$$

(Total 6 marks)

16. Consider the arithmetic sequence 8, 26, 44, ...

(a) Find an expression for the n^{th} term.

(1)

(b) Write down the sum of the first n terms using sigma notation.

(1)

(c) Calculate the sum of the first 15 terms.

(2)
(Total 4 marks)

17. (a) Simplify the difference of binomial coefficients

$$\binom{n}{3} - \binom{2n}{2}, \text{ where } n \geq 3.$$

(4)

(b) Hence, solve the inequality

$$\binom{n}{3} - \binom{2n}{2} > 32n, \text{ where } n \geq 3.$$

(2)
(Total 6 marks)

18. (a) Solve the equation $z^3 = -2 + 2\text{i}$, giving your answers in modulus–argument form.

(6)

(b) **Hence** show that one of the solutions is 1 + i when written in Cartesian form.

(1)
(Total 7 marks)

19. Given that $z = \cos\theta + \mathrm{i}\sin\theta$ show that

(a) $\mathrm{Im}\left(z^n + \dfrac{1}{z^n}\right) = 0, n \in \mathbb{Z}^+$;

(2)

(b) $\mathrm{Re}\left(\dfrac{z-1}{z+1}\right) = 0, z \neq -1$.

(5)
(Total 7 marks)

20. Expand and simplify $\left(x^2 - \dfrac{2}{x}\right)^4$.

(Total 4 marks)

21. Find the sum of all three-digit natural numbers that are not exactly divisible by 3.

(Total 5 marks)

22. Consider the complex numbers $z = 1 + 2\mathrm{i}$ and $w = 2 + a\mathrm{i}$, where $a \in \mathbb{R}$.

Find a when

(a) $|w| = 2|z|$;

(3)

(b) $\mathrm{Re}(zw) = 2\,\mathrm{Im}(zw)$.

(3)
(Total 6 marks)

23. The sum, S_n, of the first n terms of a geometric sequence, whose n^{th} term is u_n, is given by

$$S_n = \frac{7^n - a^n}{7^n}, \text{ where } a > 0.$$

(a) Find an expression for u_n.

(2)

(b) Find the first term and common ratio of the sequence.

(4)

(c) Consider the sum to infinity of the sequence.

(i) Determine the values of a such that the sum to infinity exists.

(ii) Find the sum to infinity when it exists.

(2)
(Total 8 marks)

24. Let α be the angle between the unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, where $0 \le \alpha \le \pi$.

(a) Express $|\boldsymbol{a} - \boldsymbol{b}|$ and $|\boldsymbol{a} + \boldsymbol{b}|$ in terms of α.

(3)

(b) **Hence** determine the value of $\cos\alpha$ for which $|\boldsymbol{a} + \boldsymbol{b}| = 3|\boldsymbol{a} - \boldsymbol{b}|$.

(2)
(Total 5 marks)

25. Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.

Given that $1 + \mathrm{i}$ and $1 - 2\mathrm{i}$ are zeros of $p(x)$, find the values of a, b, c and d.

(Total 7 marks)

26. The diagram below shows a solid with volume V, obtained from a cube with edge $a > 1$ when a smaller cube with edge $\frac{1}{a}$ is removed.

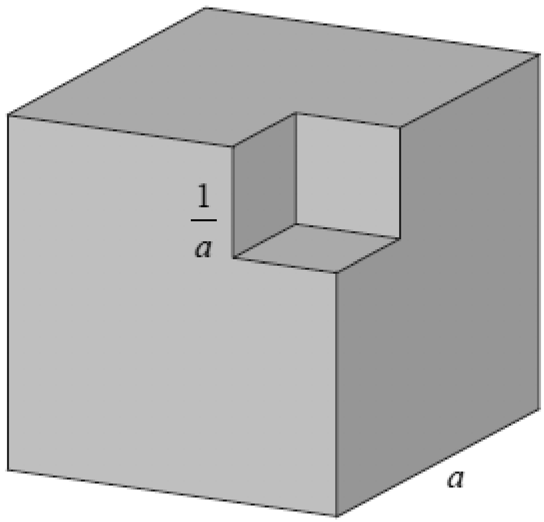

diagram not to scale

Let $x = a - \frac{1}{a}$.

(a) Find V in terms of x.

(4)

(b) Hence or otherwise, show that the only value of a for which $V = 4x$ is $a = \frac{1+\sqrt{5}}{2}$.

(4)
(Total 8 marks)

27. (a) Consider the set of numbers $a, 2a, 3a, ..., na$ where a and n are positive integers.

(i) Show that the expression for the mean of this set is $\frac{a(n+1)}{2}$.

(ii) Let $a = 4$. Find the minimum value of n for which the sum of these numbers exceeds its mean by more than 100.

(6)

(b) Consider now the set of numbers $x_1, \dots, x_m, y_1, \dots, y_1, \dots, y_n$ where $x_i = 0$ for $i = 1, \dots, m$ and $y_i = 1$ for $i = 1, \dots, n$.

(i) Show that the mean M of this set is given by $\frac{n}{m+n}$ and the standard deviation S by $\frac{\sqrt{mn}}{m+n}$.

(ii) Given that $M = S$, find the value of the median.

(11)
(Total 17 marks)

28. If z is a non-zero complex number, we define $L(z)$ by the equation

$$L(z) = \ln|z| + \mathrm{i}\arg(z),\ 0 \le \arg(z) < 2\pi.$$

(a) Show that when z is a positive real number, $L(z) = \ln z$.

(2)

(b) Use the equation to calculate

(i) $L(-1)$;

(ii) $L(1 - \mathrm{i})$;

(iii) $L(-1 + \quad \mathrm{i})$.

(5)

(c) Hence show that the property $L(z_1 z_2) = L(z_1) + L(z_2)$ does not hold for all values of z_1 and z_2.

(2)
(Total 9 marks)

29. Consider the complex number $\omega = \frac{z + \mathrm{i}}{z + 2}$, where $z = x + \mathrm{i}y$ and $\mathrm{i} = \sqrt{-1}$.

(a) If $\omega = \mathrm{i}$, determine z in the form $z = r \operatorname{cis} \theta$.

(6)

(b) Prove that $\omega = \dfrac{(x^2+2x+y^2+y)+\mathrm{i}(x+2y+2)}{(x+2)^2+y^2}$.

(3)

(c) **Hence** show that when $\mathrm{Re}(\omega) = 1$ the points (x, y) lie on a straight line, l_1, and write down its gradient.

(4)

(d) Given $\arg(z) = \arg(\omega) = \dfrac{\pi}{4}$, find $|z|$.

(6)
(Total 19 marks)

30. The mean of the first ten terms of an arithmetic sequence is 6. The mean of the first twenty terms of the arithmetic sequence is 16. Find the value of the 15^{th} term of the sequence.

(Total 6 marks)

31. Three Mathematics books, five English books, four Science books and a dictionary are to be placed on a student's shelf so that the books of each subject remain together.

(a) In how many different ways can the books be arranged?

(4)

(b) In how many of these will the dictionary be next to the Mathematics books?

(3)
(Total 7 marks)

32. Given that $z_1 = 2$ and $z_2 = 1 + \mathrm{i}\sqrt{3}$ are roots of the cubic equation $z^3 + bz^2 + cz + d = 0$ where $b, c, d \in \mathbb{R}$,

(a) write down the third root, z_3, of the equation;

(1)

(b) find the values of b, c and d;

(4)

(c) write z_2 and z_3 in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

(3)
(Total 8 marks)

33. Prove by mathematical induction $\sum_{r=1}^{n} r(r!) = (n+1)!-1,\ n \in \mathbb{Z}^{+}$.

(Total 8 marks)

34. The complex number z is defined as $z = \cos\theta + \text{i}\sin\theta$.

(a) State de Moivre’s theorem.

(1)

(b) Show that $z^n - \frac{1}{z^n} = 2\text{i}\sin(n\theta)$.

(3)

(c) Use the binomial theorem to expand $\left(z-\frac{1}{z}\right)^5$ giving your answer in simplified form.

(3)

(d) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(4)

(e) Check that your result in part (d) is true for $\theta = \frac{\pi}{4}$.

(4)

(f) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta \, \text{d}\theta$.

(4)

(g) Hence, with reference to graphs of circular functions, find $\int_0^{\frac{\pi}{2}} \cos^5\theta \, \text{d}\theta$, explaining your reasoning.

(3)
(Total 22 marks)

35. (a) Show that the complex number i is a root of the equation

$$x^4 - 5x^3 + 7x^2 - 5x + 6 = 0.$$

(2)

(b) Find the other roots of this equation.

(4)
(Total 6 marks)

36. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) Given that $\boldsymbol{X} = \boldsymbol{B} - \boldsymbol{A}^{-1}$ and $\boldsymbol{Y} = \boldsymbol{B}^{-1} - \boldsymbol{A}$,

(i) find $\boldsymbol{X}$ and $\boldsymbol{Y}$;

(ii) does $\boldsymbol{X}^{-1} + \boldsymbol{Y}^{-1}$ have an inverse? Justify your conclusion.

(5)

(b) Prove by induction that $\boldsymbol{A}^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$, for $n \in \mathbb{Z}^+$.

(7)

(c) Given that $(\boldsymbol{A}^n)^{-1} = \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$, for $n \in \mathbb{Z}^+$,

(i) find x and y in terms of n,

(ii) and hence find an expression for $\boldsymbol{A}^n + (\boldsymbol{A}^n)^{-1}$.

(6)
(Total 18 marks)

37. Six people are to sit at a circular table. Two of the people are not to sit immediately beside each other. Find the number of ways that the six people can be seated.

(Total 5 marks)

38. Consider the graphs $y = e^{-x}$ and $y = e^{-x} \sin 4x$, for $0 \le x \le \frac{5\pi}{4}$.

(a) On the same set of axes draw, on graph paper, the graphs, for $0 \le x \le \frac{5\pi}{4}$.

Use a scale of 1 cm to $\frac{\pi}{8}$ on your x-axis and 5 cm to 1 unit on your y-axis.

(3)

(b) Show that the x-intercepts of the graph $y = e^{-x} \sin 4x$ are $\frac{n\pi}{4}$, $n = 0, 1, 2, 3, 4, 5$.

(3)

(c) Find the x-coordinates of the points at which the graph of $y = e^{-x} \sin 4x$ meets the graph of $y = e^{-x}$. Give your answers in terms of π.

(3)

(d) (i) Show that when the graph of $y = e^{-x} \sin 4x$ meets the graph of $y = e^{-x}$, their gradients are equal.

(ii) Hence explain why these three meeting points are not local maxima of the graph $y = e^{-x} \sin 4x$.

(6)

(e) (i) Determine the y-coordinates, y_1, y_2 and y_3, where $y_1 > y_2 > y_3$, of the local maxima of $y = e^{-x} \sin 4x$ for $0 \le x \le \frac{5\pi}{4}$. You do not need to show that they are maximum values, but the values should be simplified.

(ii) Show that y_1, y_2 and y_3 form a geometric sequence and determine the common ratio r.

(7)
(Total 22 marks)

39. Find the values of n such that $(1 + \sqrt{3}\,\text{i})^n$ is a real number.

(Total 5 marks)

40. (a) The sum of the first six terms of an arithmetic series is 81. The sum of its first eleven terms is 231. Find the first term and the common difference.

(6)

(b) The sum of the first two terms of a geometric series is 1 and the sum of its first four terms is 5. If all of its terms are positive, find the first term and the common ratio.

(5)

(c) The r^{th} term of a new series is defined as the product of the r^{th} term of the arithmetic series and the r^{th} term of the geometric series above. Show that the r^{th} term of this new series is $(r + 1)2^{r-1}$.

(3)

(d) Using mathematical induction, prove that

$$\sum_{r=1}^{n} (r+1)2^{r-1} = n2^n, n \in \mathbb{Z}^+.$$

(7)
(Total 21 marks)

41. (a) Let $z = x + \mathrm{i}y$ be any non-zero complex number.

(i) Express $\frac{1}{z}$ in the form $u + \mathrm{i}v$.

(ii) If $z + \frac{1}{z} = k, k \in \mathbb{R}$, show that either $y = 0$ or $x^2 + y^2 = 1$.

(iii) Show that if $x^2 + y^2 = 1$ then $|k| \leq 2$.

(8)

(b) Let $w = \cos\theta + \mathrm{i}\sin\theta$.

(i) Show that $w^n + w^{-n} = 2\cos n\theta, n \in \mathbb{Z}$.

(ii) Solve the equation $3w^2 - w + 2 - w^{-1} + 3w^{-2} = 0$, giving the roots in the form $x + \mathrm{i}y$.

(14)
(Total 22 marks)

42. When $\left(1+\frac{x}{2}\right)^n, n \in \mathbb{N}$, is expanded in ascending powers of x, the coefficient of x^3 is 70.

(a) Find the value of n.

(5)

(b) Hence, find the coefficient of x^2.

(1)
(Total 6 marks)

43. Consider the equation $z^3 + az^2 + bz + c = 0$, where $a, b, c \in \mathbb{R}$. The points in the Argand diagram representing the three roots of the equation form the vertices of a triangle whose area is 9. Given that one root is $-1 + 3\mathrm{i}$, find

(a) the other two roots;

(4)

(b) a, b and c.

(3)
(Total 7 marks)

44. Express $\dfrac{1}{\left(1-\mathrm{i}\sqrt{3}\right)^3}$ in the form $\dfrac{a}{b}$ where $a, b \in \mathbb{Z}$.

(Total 5 marks)

45. A circular disc is cut into twelve sectors whose areas are in an arithmetic sequence.
The angle of the largest sector is twice the angle of the smallest sector.

Find the size of the angle of the smallest sector.

(Total 5 marks)

46. The common ratio of the terms in a geometric series is 2^x.

(a) State the set of values of x for which the sum to infinity of the series exists.

(2)

(b) If the first term of the series is 35, find the value of x for which the sum to infinity is 40.

(4)
(Total 6 marks)

47. The function f is defined by $f(x) = x\,\mathrm{e}^{2x}$.

It can be shown that $f^{(n)}(x) = (2^n x + n\,2^{n-1})\,\mathrm{e}^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

(a) By considering $f^{(n)}(x)$ for $n = 1$ and $n = 2$, show that there is one minimum point P on the graph of f, and find the coordinates of P.

(7)

(b) Show that f has a point of inflexion Q at $x = -1$.

(5)

(c) Determine the intervals on the domain of f where f is

(i) concave up;

(ii) concave down.

(2)

(d) Sketch f, clearly showing any intercepts, asymptotes and the points P and Q.

(4)

(e) Use mathematical induction to prove that $f^{(n)}(x) = (2^n x + n2^{n-1})\,e^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

(9)
(Total 27 marks)

48. (a) Find the sum of the infinite geometric sequence 27, −9, 3, −1, ...

(3)

(b) Use mathematical induction to prove that for $n \in \mathbb{Z}^+$,

$$a + ar + ar^2 + ... + ar^{n-1} = \frac{a\left(1-r^n\right)}{1-r}.$$

(7)
(Total 10 marks)

49. Let $w = \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$.

(a) Show that w is a root of the equation $z^5 - 1 = 0$.

(3)

(b) Show that $(w - 1)(w^4 + w^3 + w^2 + w + 1) = w^5 - 1$ and deduce that $w^4 + w^3 + w^2 + w + 1 = 0$.

(3)

(c) **Hence** show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

(6)
(Total 12 marks)

50. $z_1 = \left(1 + i\sqrt{3}\right)^m$ and $z_2 = (1 - i)^n$.

(a) Find the modulus and argument of z_1 and z_2 in terms of m and n, respectively.

(6)

(b) **Hence**, find the smallest positive integers m and n such that $z_1 = z_2$.

(8)
(Total 14 marks)

51. Determine the first three terms in the expansion of $(1 - 2x)^5 (1 + x)^7$ in ascending powers of x.

(Total 5 marks)

52. Find, in its simplest form, the argument of $(\sin\theta + i\,(1 - \cos\theta))^2$ where θ is an acute angle.

(Total 7 marks)

53. Consider $w = \dfrac{z}{z^2 + 1}$ where $z = x + iy$, $y \neq 0$ and $z^2 + 1 \neq 0$.

Given that Im $w = 0$, show that $|z| = 1$.

(Total 7 marks)

54. Let $\boldsymbol{M}^2 = \boldsymbol{M}$ where $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $bc \neq 0$.

(a) (i) Show that $a + d = 1$.

(ii) Find an expression for bc in terms of a.

(5)

(b) **Hence** show that $\boldsymbol{M}$ is a singular matrix.

(3)

(c) If all of the elements of $\boldsymbol{M}$ are positive, find the range of possible values for a.

(3)

(d) Show that $(\boldsymbol{I} - \boldsymbol{M})^2 = \boldsymbol{I} - \boldsymbol{M}$ where $\boldsymbol{I}$ is the identity matrix.

(3)

(e) Prove by mathematical induction that $(\boldsymbol{I} - \boldsymbol{M})^n = \boldsymbol{I} - \boldsymbol{M}$ for $n \in \mathbb{Z}^+$.

(6)
(Total 20 marks)

55. (a) The independent random variables X and Y have Poisson distributions and $Z = X + Y$. The means of X and Y are λ and μ respectively. By using the identity

$$\mathrm{P}(Z = n) = \sum_{k=0}^{n} \mathrm{P}(X = k)\,\mathrm{P}(Y = n - k)$$

show that Z has a Poisson distribution with mean $(\lambda + \mu)$.

(6)

(b) Given that U_1, U_2, U_3, … are independent Poisson random variables each having mean m, use mathematical induction together with the result in (a) to show that $\sum_{r=1}^{n} U_r$ has a Poisson distribution with mean nm.

(6)
(Total 12 marks)

56. An 81 metre rope is cut into n pieces of increasing lengths that form an arithmetic sequence with a common difference of d metres. Given that the lengths of the shortest and longest pieces are 1.5 metres and 7.5 metres respectively, find the values of n and d.

(Total 4 marks)

57. (a) Using mathematical induction, prove that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix},\ n \in \mathbb{Z}^+.$$

(9)

(b) Show that the result holds true for $n = -1$.

(5)
(Total 14 marks)

58. (a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \mathrm{i}$.

(6)

(b) Draw these roots on an Argand diagram.

(2)

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\frac{z_2}{z_1}$ in the form $a + \mathrm{i}b$.

(4)
(Total 12 marks)

59. (a) Expand and simplify $(x - 1)(x^4 + x^3 + x^2 + x + 1)$.

(2)

(b) Given that b is a root of the equation $z^5 - 1 = 0$ which does not lie on the real axis in the Argand diagram, show that $1 + b + b^2 + b^3 + b^4 = 0$.

(3)

(c) If $u = b + b^4$ and $v = b^2 + b^3$ show that

(i) $u + v = uv = -1$;

(ii) $u - v = \sqrt{5}$, given that $u - v > 0$.

(8)
(Total 13 marks)

60. A geometric sequence has a first term of 2 and a common ratio of 1.05. Find the value of the smallest term that is greater than 500.

(Total 5 marks)

61. There are six boys and five girls in a school tennis club. A team of two boys and two girls will be selected to represent the school in a tennis competition.

(a) In how many different ways can the team be selected?

(3)

(b) Tim is the youngest boy in the club and Anna is the youngest girl. In how many different ways can the team be selected if it must include both of them?

(2)

(c) What is the probability that the team includes both Tim and Anna?

(1)

(d) Fred is the oldest boy in the club. Given that Fred is selected for the team, what is the probability that the team includes Tim or Anna, but not both?

(4)
(Total 10 marks)

62. Given that $4 \ln 2 - 3\ln 4 = -\ln k$, find the value of k.

(Total 5 marks)

63. Solve the equation $\log_3(x + 17) - 2 = \log_3 2x$.

(Total 5 marks)

64. Solve the equation $2^{2x+2} - 10 \times 2^x + 4 = 0$, $x \in \mathbb{R}$.

(Total 6 marks)

65. Given that $(a + b\text{i})^2 = 3 + 4\text{i}$ obtain a pair of simultaneous equations involving a and b. Hence find the two square roots of $3 + 4\text{i}$.

(Total 7 marks)

66. Given that $2 + \text{i}$ is a root of the equation $x^3 - 6x^2 + 13x - 10 = 0$ find the other two roots.

(Total 5 marks)

67. Given that $|z| = \sqrt{10}$, solve the equation $5z + \dfrac{10}{z^*} = 6 - 18\text{i}$, where z^* is the conjugate of z.

(Total 7 marks)

68. Find the three cube roots of the complex number 8i. Give your answers in the form $x + \text{i}y$.

(Total 8 marks)

69. Solve the simultaneous equations

$$\begin{aligned} \text{i}z_1 + 2z_2 &= 3 \\ z_1 + (1 - \text{i})z_2 &= 4 \end{aligned}$$

giving z_1 and z_2 in the form $x + \text{i}y$, where x and y are real.

(Total 9 marks)

70. Find b where $\dfrac{2+b\mathrm{i}}{1-b\mathrm{i}} = \dfrac{7}{10} + \dfrac{9}{10}\mathrm{i}$.

(Total 6 marks)

71. Use mathematical induction to prove that $5^n + 9^n + 2$ is divisible by 4, for $n \in \mathbb{Z}^+$.

(Total 9 marks)

72. Given that $z = (b + \mathrm{i})^2$, where b is real and positive, find the value of b when $\arg z = 60°$.

(Total 6 marks)

73. A room has nine desks arranged in three rows of three desks. Three students sit in the room. If the students randomly choose a desk find the probability that two out of the front three desks are chosen.

(Total 5 marks)

74. (a) Show that $p = 2$ is a solution to the equation $p^3 + p^2 – 5p – 2 = 0$.

(2)

(b) Find the values of a and b such that $p^3 + p^2 – 5p – 2 = (p – 2)(p^2 + ap + b)$.

(4)

(c) Hence find the other two roots to the equation $p^3 + p^2 – 5p – 2 = 0$.

(3)

(d) An arithmetic sequence has p as its common difference. Also, a geometric sequence has p as its common ratio. Both sequences have 1 as their first term.

(i) Write down, in terms of p, the first four terms of each sequence.

(ii) If the sum of the third and fourth terms of the arithmetic sequence is equal to the sum of the third and fourth terms of the geometric sequence, find the three possible values of p.

(iii) For which value of p found in (d)(ii) does the sum to infinity of the terms of the geometric sequence exist?

(iv) For the same value p, find the sum of the first 20 terms of the arithmetic sequence, writing your answer in the form $a + b\sqrt{c}$, where $a, b, c \in \mathbb{Z}$.

(13)
(Total 22 marks)

75. Consider the complex geometric series $e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + ...$

(a) Find an expression for z, the common ratio of this series.

(2)

(b) Show that $|z| < 1$.

(2)

(c) Write down an expression for the sum to infinity of this series.

(2)

(d) (i) Express your answer to part (c) in terms of $\sin\theta$ and $\cos\theta$.

(ii) Hence show that

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + ... = \frac{4\cos\theta - 2}{5 - 4\cos\theta}.$$

(10)
(Total 16 marks)

76. The roots of the equation $z^2 + 2z + 4 = 0$ are denoted by α and β?

(a) Find α and β in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

(6)

(b) Given that α lies in the second quadrant of the Argand diagram, mark α and β on an Argand diagram.

(2)

(c) Use the principle of mathematical induction to prove De Moivre's theorem, which states that $\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n$ for $n \in \mathbb{Z}^+$.

(8)

(d) Using De Moivre's theorem find $\dfrac{\alpha^3}{\beta^2}$ in the form $a + \mathrm{i}b$.

(4)

(e) Using De Moivre's theorem or otherwise, show that $\alpha^3 = \beta^3$.

(3)

(f) Find the exact value of $\alpha\beta^* + \beta\alpha^*$ where α^* is the conjugate of α and β^* is the conjugate of β.

(5)

(g) Find the set of values of n for which α^n is real.

(3)
(Total 31 marks)

77. A sum of \$ 5000 is invested at a compound interest rate of 6.3 % per annum.

(a) Write down an expression for the value of the investment after n full years.

(1)

(b) What will be the value of the investment at the end of five years?

(1)

(c) The value of the investment will exceed \$10 000 after n full years.

(i) Write an inequality to represent this information.

(ii) Calculate the minimum value of n.

(4)
(Total 6 marks)