1. Find the area enclosed by the curve $y = \arctan x$, the x-axis and the line $x = \sqrt{3}$.

(Total 6 marks)

2. Show that the points (0, 0) and $(\sqrt{2\pi}, -\sqrt{2\pi})$ on the curve $e^{(x+y)} = \cos(xy)$ have a common tangent.

(Total 7 marks)

3. Consider the function $f(x) = \frac{\ln x}{x}$, $0 < x < e^2$.

(a) (i) Solve the equation $f'(x) = 0$.

(ii) Hence show the graph of f has a local maximum.

(iii) Write down the range of the function f.

(5)

(b) Show that there is a point of inflexion on the graph and determine its coordinates.

(5)

(c) Sketch the graph of $y = f(x)$, indicating clearly the asymptote, x-intercept and the local maximum.

(3)

(d) Now consider the functions $g(x) = \frac{\ln|x|}{x}$ and $h(x) = \frac{\ln|x|}{|x|}$, where $0 < |x| < e^2$.

(i) Sketch the graph of $y = g(x)$.

(ii) Write down the range of g.

(iii) Find the values of x such that $h(x) > g(x)$.

(6)

(Total 19 marks)

4. The quadratic function $f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$.

(a) Find the value of p and the value of q.

(4)

(b) The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph.

(2)
(Total 6 marks)

5. The curve C has equation $y = \frac{1}{8}(9 + 8x^2 - x^4)$.

(a) Find the coordinates of the points on C at which $\frac{dy}{dx} = 0$.

(4)

(b) The tangent to C at the point P(1, 2) cuts the x-axis at the point T. Determine the coordinates of T.

(4)

(c) The normal to C at the point P cuts the y-axis at the point N. Find the area of triangle PTN.

(7)
(Total 15 marks)

6. (a) (i) Sketch the graphs of $y = \sin x$ and $y = \sin 2x$, on the same set of axes, for $0 \le x \le \frac{\pi}{2}$.

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain $0 \le x \le \frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs.

(9)

(b) Find the value of $\int_0^1 \sqrt{\frac{x}{4-x}}\,\mathrm{d}x$ using the substitution $x = 4\sin^2\theta$.

(8)

(c) The increasing function f satisfies $f(0) = 0$ and $f(a) = b$, where $a > 0$ and $b > 0$.

(i) By reference to a sketch, show that $\int_0^a f(x)\,\mathrm{d}x = ab - \int_0^b f^{-1}(x)\,\mathrm{d}x$.

(ii) **Hence** find the value of $\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x$.

(8)
(Total 25 marks)

7. Consider the function $f(x) = x^3 - 3x^2 - 9x + 10,\ x \in \mathbb{R}$.

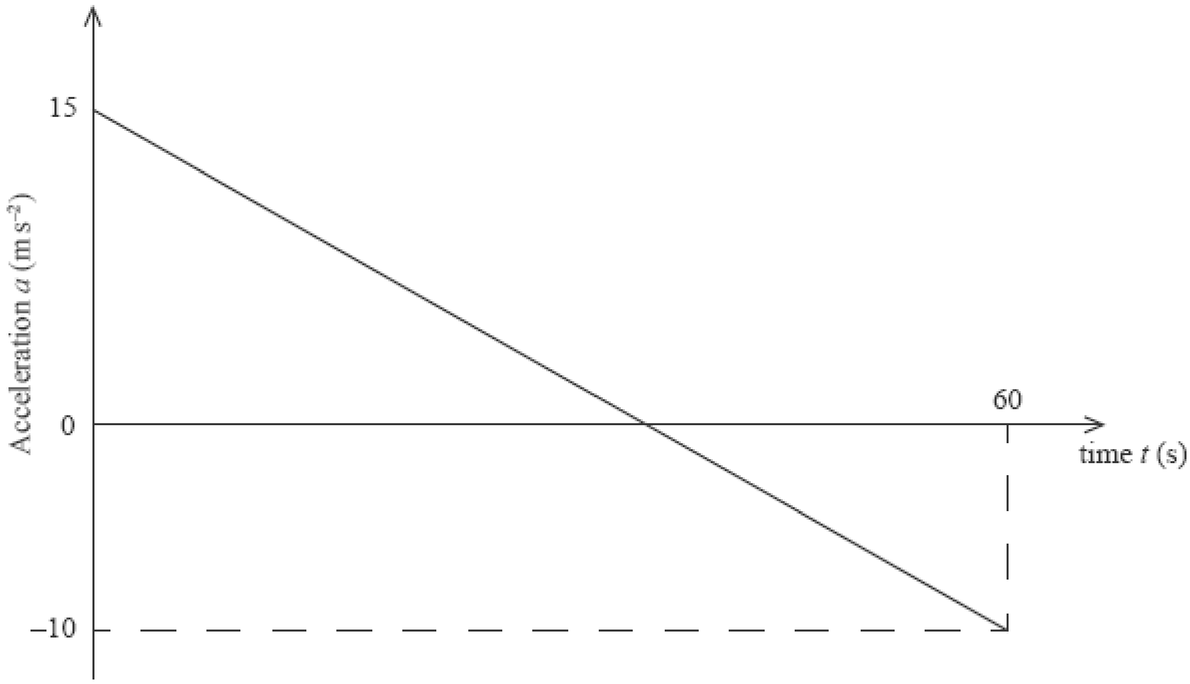
(a) Find the equation of the straight line passing through the maximum and minimum points of the graph $y = f(x)$.

(4)

(b) Show that the point of inflexion of the graph $y = f(x)$ lies on this straight line.

(2)
(Total 6 marks)

8. A jet plane travels horizontally along a straight path for one minute, starting at time $t = 0$, where t is measured in seconds. The acceleration, a, measured in m s^{-2}, of the jet plane is given by the straight line graph below.



(a) Find an expression for the acceleration of the jet plane during this time, in terms of t.

(1)

(b) Given that when $t = 0$ the jet plane is travelling at 125 m s^{-1}, find its maximum velocity in m s^{-1} during the minute that follows.

(4)

(c) Given that the jet plane breaks the sound barrier at 295 m s^{-1}, find out for how long the jet plane is travelling greater than this speed.

(3)
(Total 8 marks)

9. An open glass is created by rotating the curve $y = x^2$, defined in the domain $x \in [0, 10]$, 2π radians about the y-axis. Units on the coordinate axes are defined to be in centimetres.

(a) When the glass contains water to a height h cm, find the volume V of water in terms of h.

(3)

(b) If the water in the glass evaporates at the rate of 3 cm^3 per hour for each cm^2 of exposed surface area of the water, show that,

$$\frac{dV}{dt} = -3\sqrt{2\pi V}\text{, where } t \text{ is measured in hours.}$$

(6)

(c) If the glass is filled completely, how long will it take for all the water to evaporate?

(7)
(Total 16 marks)

10. A skydiver jumps from a stationary balloon at a height of 2000 m above the ground. Her velocity, v m s^{-1}, t seconds after jumping, is given by $v = 50(1 - e^{-0.2t})$.

(a) Find her acceleration 10 seconds after jumping.

(3)

(b) How far above the ground is she 10 seconds after jumping?

(3)
(Total 6 marks)

11. A rocket is rising vertically at a speed of 300 m s^{-1} when it is 800 m directly above the launch site. Calculate the rate of change of the distance between the rocket and an observer, who is 600 m from the launch site and on the same horizontal level as the launch site.

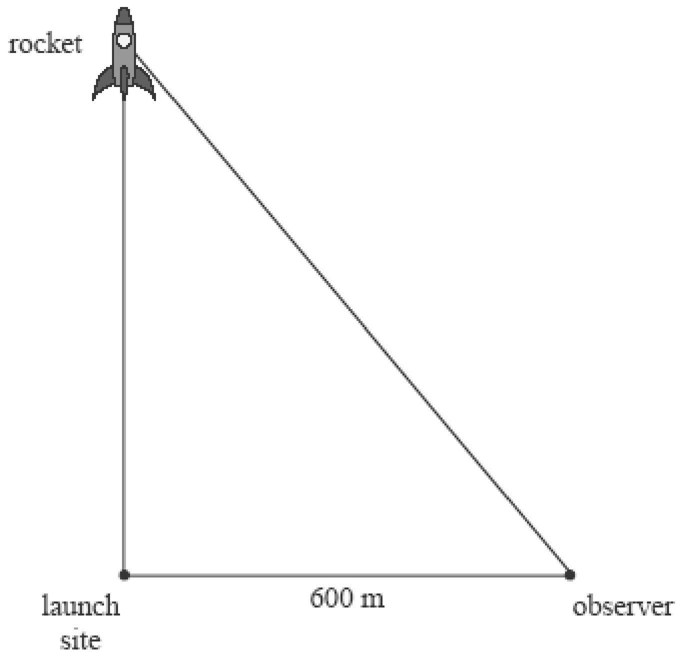


diagram not to scale

(Total 6 marks)

12. The point P, with coordinates (p, q), lies on the graph of $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, $a > 0$.
The tangent to the curve at P cuts the axes at $(0, m)$ and $(n, 0)$. Show that $m + n = a$.

(Total 8 marks)

13. (a) Using integration by parts, show that $\int e^{2x} \sin x \, dx = \frac{1}{5} e^{2x}(2\sin x - \cos x) + C$.

(6)

(b) Solve the differential equation $\frac{dy}{dx} = \sqrt{1 - y^2}\, e^{2x} \sin x$, given that $y = 0$ when $x = 0$, writing your answer in the form $y = f(x)$.

(5)

(c) (i) Sketch the graph of $y = f(x)$, found in part (b), for $0 \le x \le 1.5$.
Determine the coordinates of the point P, the first positive intercept on the x-axis, and mark it on your sketch.

(ii) The region bounded by the graph of $y = f(x)$ and the x-axis, between the origin and P, is rotated 360° about the x-axis to form a solid of revolution.
Calculate the volume of this solid.

(6)
(Total 17 marks)

14. The region enclosed between the curves $y = \sqrt{x}e^x$ and $y = e\sqrt{x}$ is rotated through 2π about the x-axis. Find the volume of the solid obtained.

(Total 7 marks)

15. (a) Given that $\alpha > 1$, use the substitution $u = \frac{1}{x}$ to show that

$$\int_1^{\alpha} \frac{1}{1+x^2}\mathrm{d}x = \int_{\frac{1}{\alpha}}^{1} \frac{1}{1+u^2}\mathrm{d}u .$$

(5)

(b) **Hence** show that $\arctan \alpha + \arctan\frac{1}{\alpha} = \frac{\pi}{2}$.

(2)
(Total 7 marks)

16. Consider $f(x) = \frac{x^2 - 5x + 4}{x^2 + 5x + 4}$.

(a) Find the equations of all asymptotes of the graph of f.

(4)

(b) Find the coordinates of the points where the graph of f meets the x and y axes.

(2)

(c) Find the coordinates of

(i) the maximum point and justify your answer;

(ii) the minimum point and justify your answer.

(10)

(d) Sketch the graph of f, clearly showing all the features found above.

(3)

(e) **Hence**, write down the number of points of inflexion of the graph of f.

(1)
(Total 20 marks)

17. The function f is defined by $f(x) = \mathrm{e}^{x^2-2x-1.5}$.

(a) Find $f'(x)$.

(2)

(b) You are given that $y = \dfrac{f(x)}{x-1}$ has a local minimum at $x = a$, $a > 1$. Find the value of a.

(6)
(Total 8 marks)

18. The normal to the curve $x\mathrm{e}^{-y} + \mathrm{e}^{y} = 1 + x$, at the point $(c, \ln c)$, has a y-intercept $c^2 + 1$.

Determine the value of c.

(Total 7 marks)

19. Find the value of $\int_0^1 t\ln(t+1)\mathrm{d}t$.

(Total 6 marks)

20. Throughout this question x satisfies $0 \le x < \frac{\pi}{2}$.

(a) Solve the differential equation $\sec^2 x \frac{\mathrm{d}y}{\mathrm{d}x} = -y^2$, where $y = 1$ when $x = 0$.

Give your answer in the form $y = f(x)$.

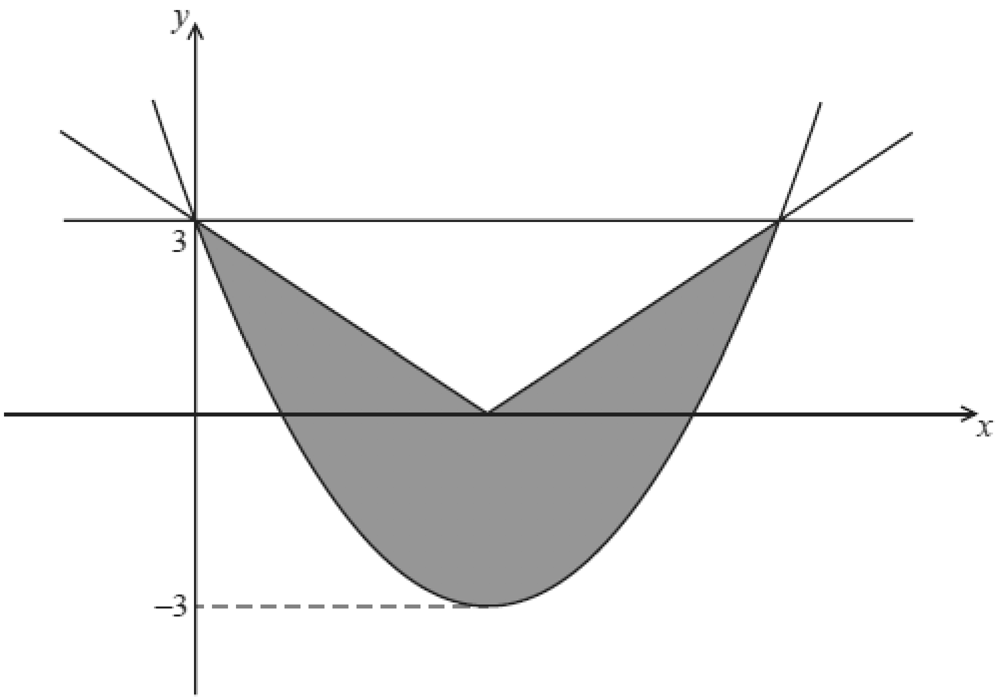
(7)

(b) (i) Prove that $1 \le \sec x \le 1 + \tan x$.

(ii) Deduce that $\frac{\pi}{4} \le \int_0^{\frac{\pi}{4}} \sec x\mathrm{d}x \le \frac{\pi}{4} + \frac{1}{2}\ln 2$.

(8)
(Total 15 marks)

21. The diagram below shows the graphs of $y = \left|\frac{3}{2}x - 3\right|$, $y = 3$ and a quadratic function, that all intersect in the same two points.



Given that the minimum value of the quadratic function is –3, find an expression for the area of the shaded region in the form $\int_0^t (ax^2 + bx + c)\,\mathrm{d}x$, where the constants a, b, c and t are to be determined. (Note: The integral does not need to be evaluated.)

(Total 8 marks)

22. Points A, B and C are on the circumference of a circle, centre O and radius r.
A trapezium OABC is formed such that AB is parallel to OC, and the angle $A\hat{O}C$ is θ, $\frac{\pi}{2} \le \theta < \pi$.

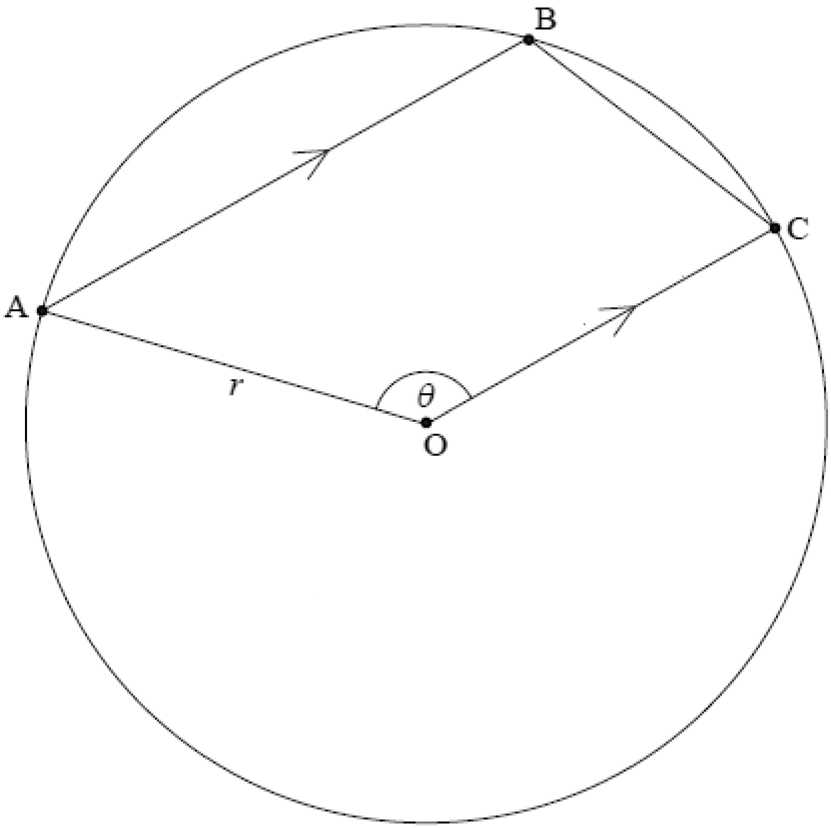


diagram not to scale

(a) Show that angle $B\hat{O}C$ is $\pi - \theta$.

(3)

(b) Show that the area, T, of the trapezium can be expressed as

$$T = \frac{1}{2}r^2 \sin\theta - \frac{1}{2}r^2 \sin 2\theta\ .$$

(3)

(c) (i) Show that when the area is maximum, the value of θ satisfies

$$\cos\theta = 2\cos 2\theta.$$

(ii) **Hence** determine the maximum area of the trapezium when $r = 1$.
(Note: It is not required to prove that it is a maximum.)

(5)
(Total 11 marks)

23. A body is moving through a liquid so that its acceleration can be expressed as

$$\left(-\frac{v^2}{200} - 32\right) \text{ m s}^{-2},$$

where v m s^{-1} is the velocity of the body at time t seconds.

The initial velocity of the body was known to be 40 m s^{-1}.

(a) Show that the time taken, T seconds, for the body to slow to V m s^{-1} is given by

$$T = 200\int_V^{40} \frac{1}{v^2 + 80^2}\,\mathrm{d}v.$$

(4)

(b) (i) Explain why acceleration can be expressed as $v\dfrac{\mathrm{d}v}{\mathrm{d}s}$, where s is displacement, in metres, of the body at time t seconds.

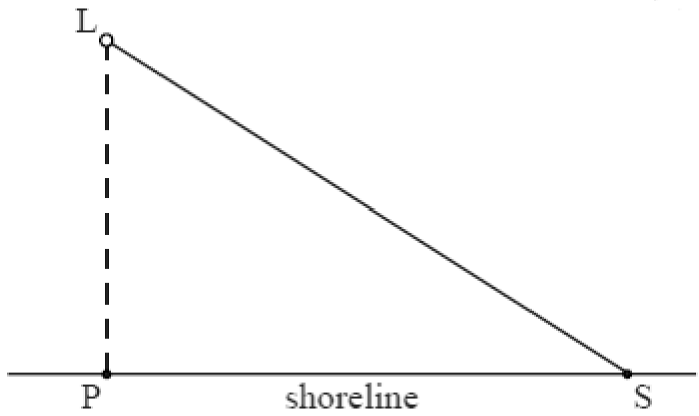
(ii) **Hence** find a similar integral to that shown in part (a) for the distance, S metres, travelled as the body slows to V m s^{-1}.

(7)

(c) **Hence**, using parts (a) and (b), find the distance travelled and the time taken until the body momentarily comes to rest.

(3)
(Total 14 marks)

24. A lighthouse L is located offshore, 500 metres from the nearest point P on a long straight shoreline. The narrow beam of light from the lighthouse rotates at a constant rate of 8π radians per minute, producing an illuminated spot S that moves along the shoreline. You may assume that the height of the lighthouse can be ignored and that the beam of light lies in the horizontal plane defined by sea level.



When S is 2000 metres from P,

(a) show that the speed of S, correct to three significant figures, is 214 000 metres per minute;

(5)

(b) find the acceleration of S.

(3)
(Total 8 marks)

25. The function f is defined by

$$f(x) = (x^3 + 6x^2 + 3x - 10)^{\frac{1}{2}}, \text{ for } x \in D,$$

where $D \subseteq \mathbb{R}$ is the greatest possible domain of f.

(a) Find the roots of $f(x) = 0$.

(2)

(b) Hence specify the set D.

(2)

(c) Find the coordinates of the local maximum on the graph $y = f(x)$.

(2)

(d) Solve the equation $f(x) = 3$.

(2)

(e) Sketch the graph of $|y| = f(x)$, for $x \in D$.

(3)

(f) Find the area of the region completely enclosed by the graph of $|y| = f(x)$.

(3)
(Total 14 marks)

26. The functions f, g and h are defined by

$$f(x) = 1 + e^x, \text{ for } x \in \mathbb{R},$$
$$g(x) = \frac{1}{x}, \text{ for } x \in \mathbb{R}/\{0\},$$
$$h(x) = \sec x, \text{ for } x \in \mathbb{R}/\left\{\frac{2n+1}{2}\pi, n \in \mathbb{Z}\right\}.$$

(a) Determine the range of the composite function $g \circ f$.

(3)

(b) Determine the inverse of the function $g \circ f$, clearly stating the domain.

(4)

(c) (i) Show that the function $y = (f \circ g \circ h)(x)$ satisfies the differential equation

$$\frac{dy}{dx} = (1 - y)\sin x.$$

(ii) Hence, or otherwise, find $\int y \sin x dx$, as a function of x.

(iii) You are given that the domain of $y = (f \circ g \circ h)(x)$ can be extended to the whole real axis. That part of the graph of $y = (f \circ g \circ h)(x)$, between its maximum at $x = 0$ and its first minimum for positive x, is rotated by 2π about the y-axis. Calculate the volume of the solid generated.

(14)
(Total 21 marks)

27. Find y in terms of x, given that $(1 + x^3)\frac{dy}{dx} = 2x^2 \tan y$ and $y = \frac{\pi}{2}$ when $x = 0$.

(Total 7 marks)

28. Consider the curve $y = xe^x$ and the line $y = kx$, $k \in \mathbb{R}$.

(a) Let $k = 0$.

(i) Show that the curve and the line intersect once.

(ii) Find the angle between the tangent to the curve and the line at the point of intersection.

(5)

(b) Let $k = 1$. Show that the line is a tangent to the curve.

(3)

(c) (i) Find the values of k for which the curve $y = xe^x$ and the line $y = kx$ meet in two distinct points.

(ii) Write down the coordinates of the points of intersection.

(iii) Write down an integral representing the area of the region A enclosed by the curve and the line.

(iv) **Hence**, given that $0 < k < 1$, show that $A < 1$.

(15)
(Total 23 marks)

29. Find the equation of the normal to the curve $x^3y^3 - xy = 0$ at the point (1, 1).

(Total 7 marks)

30. The line $y = m(x - m)$ is a tangent to the curve $(1 - x)y = 1$.

Determine m and the coordinates of the point where the tangent meets the curve.

(Total 7 marks)

31. Let $f(x) = \dfrac{a + be^x}{ae^x + b}$, where $0 < b < a$.

(a) Show that $f'(x) = \dfrac{(b^2 - a^2)e^x}{(ae^x + b)^2}$.

(3)

(b) **Hence** justify that the graph of f has no local maxima or minima.

(2)

(c) Given that the graph of f has a point of inflexion, find its coordinates.

(6)

(d) Show that the graph of f has exactly two asymptotes.

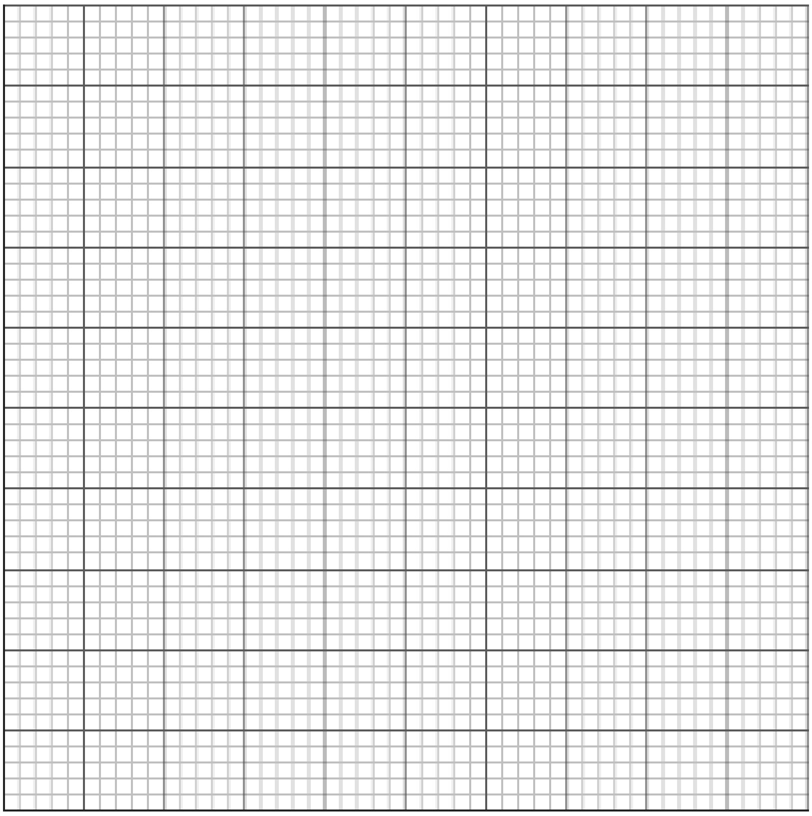
(3)

(e) Let $a = 4$ and $b = 1$. Consider the region R enclosed by the graph of $y = f(x)$, the y-axis and the line with equation $y = \dfrac{1}{2}$.

Find the volume V of the solid obtained when R is rotated through 2π about the x-axis.

(5)
(Total 19 marks)

32. (a) Let $a > 0$. Draw the graph of $y = \left|x - \frac{a}{2}\right|$ for $-a \le x \le a$ on the grid below.



(2)

(b) Find k such that $\int_{-a}^{0} \left|x - \frac{a}{2}\right| \mathrm{d}x = k \int_{0}^{a} \left|x - \frac{a}{2}\right| \mathrm{d}x$.

(5)
(Total 7 marks)

33. Let f be a function defined by $f(x) = x - \arctan x$, $x \in \mathbb{R}$.

(a) Find $f(1)$ and $f(-\sqrt{3})$.

(2)

(b) Show that $f(-x) = -f(x)$, for $x \in \mathbb{R}$.

(2)

(c) Show that $x - \frac{\pi}{2} < f(x) < x + \frac{\pi}{2}$, for $x \in \mathbb{R}$.

(2)

(d) Find expressions for $f'(x)$ and $f''(x)$. Hence describe the behaviour of the graph of f at the origin and justify your answer.

(8)

(e) Sketch a graph of f, showing clearly the asymptotes.

(3)

(f) Justify that the inverse of f is defined for all $x \in \mathbb{R}$ and sketch its graph.

(3)
(Total 20 marks)

34. Let f be a function with domain $\mathbb{R}$ that satisfies the conditions,

$$f(x + y) = f(x)\,f(y), \text{ for all } x \text{ and } y \text{ and } f(0) \neq 0.$$

(a) Show that $f(0) = 1$.

(3)

(b) Prove that $f(x) \neq 0$, for all $x \in \mathbb{R}$.

(3)

(c) Assuming that $f'(x)$ exists for all $x \in \mathbb{R}$, use the definition of derivative to show that $f(x)$ satisfies the differential equation $f'(x) = k\,f(x)$, where $k = f'(0)$.

(4)

(d) Solve the differential equation to find an expression for $f(x)$.

(4)
(Total 14 marks)

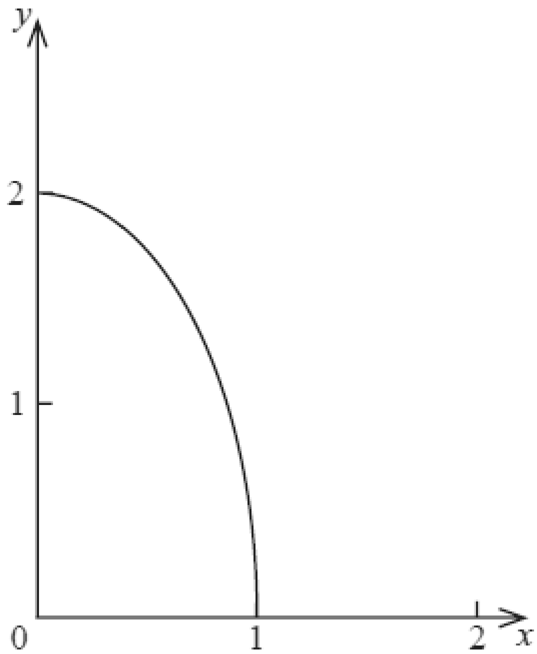
35. (a) Show that $\frac{3}{x+1} + \frac{2}{x+3} = \frac{5x+11}{x^2+4x+3}$.

(2)

(b) Hence find the value of k such that $\int_0^2 \frac{5x+11}{x^2+4x+3}\,\mathrm{d}x = \ln k$.

(4)
(Total 6 marks)

36. Consider the part of the curve $4x^2 + y^2 = 4$ shown in the diagram below.



(a) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

(3)

(b) Find the gradient of the tangent at the point $\left(\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

(1)

(c) A bowl is formed by rotating this curve through 2π radians about the x-axis. Calculate the volume of this bowl.

(4)
(Total 8 marks)

37. A function is defined as $f(x) = k\sqrt{x}$, with $k > 0$ and $x \geq 0$.

(a) Sketch the graph of $y = f(x)$.

(1)

(b) Show that f is a one-to-one function.

(1)

(c) Find the inverse function, $f^{-1}(x)$ and state its domain.

(3)

(d) If the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at the point (4, 4) find the value of k.

(2)

(e) Consider the graphs of $y = f(x)$ and $y = f^{-1}(x)$ using the value of k found in part (d).

(i) Find the area enclosed by the two graphs.

(ii) The line $x = c$ cuts the graphs of $y = f(x)$ and $y = f^{-1}(x)$ at the points P and Q respectively. Given that the tangent to $y = f(x)$ at point P is parallel to the tangent to $y = f^{-1}(x)$ at point Q find the value of c.

(9)
(Total 16 marks)

38. Let $f(x) = \frac{1-x}{1+x}$ and $g(x) = \sqrt{x+1}$, $x > -1$.

Find the set of values of x for which $f'(x) \le f(x) \le g(x)$.

(Total 7 marks)

39. (a) Integrate $\int \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta$.

(3)

(b) Given that $\int_{\frac{\pi}{2}}^{a} \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta = \frac{1}{2}$ and $\frac{\pi}{2} < a < \pi$, find the value of a.

(2)
(Total 5 marks)

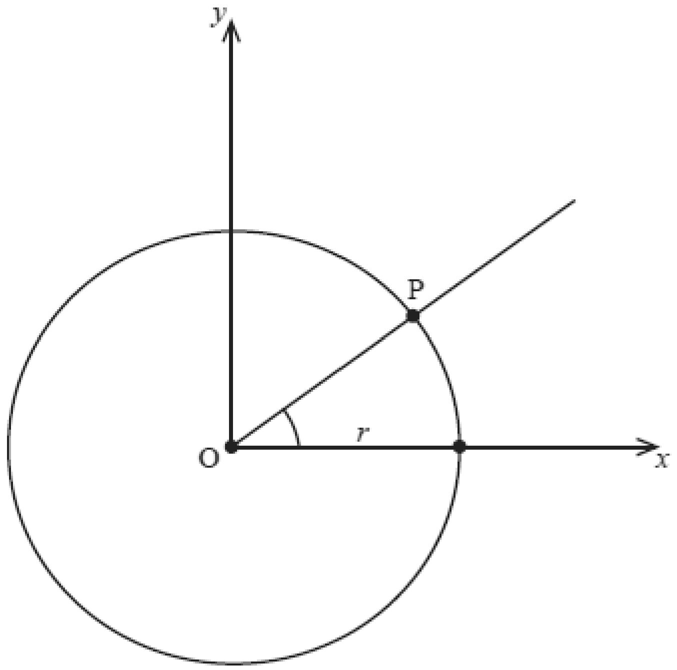
40. (a) Solve the differential equation $\frac{\cos^2 x}{\mathrm{e}^y} - \mathrm{e}^{\mathrm{e}^y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, given that $y = 0$ when $x = \pi$.

(7)

(b) Find the value of y when $x = \frac{\pi}{2}$.

(1)
(Total 8 marks)

41. The diagram below shows a circle with centre at the origin O and radius $r > 0$.



A point P(x, y), ($x > 0$, $y > 0$) is moving round the circumference of the circle.

Let $m = \tan\left(\arcsin\frac{y}{r}\right)$.

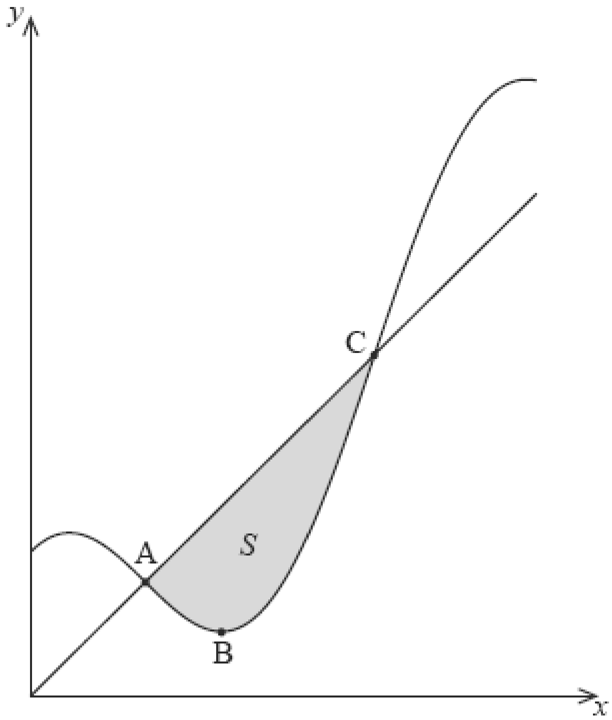
(a) Given that $\frac{\mathrm{d}y}{\mathrm{d}t} = 0.001r$, show that $\frac{\mathrm{d}m}{\mathrm{d}t} = \left(\frac{r}{10\sqrt{r^2 - y^2}}\right)^3$.

(6)

(b) State the geometrical meaning of $\frac{\mathrm{d}m}{\mathrm{d}t}$.

(1)
(Total 7 marks)

42. Let f be a function defined by $f(x) = x + 2\cos x$, $x \in [0, 2\pi]$. The diagram below shows a region S bound by the graph of f and the line $y = x$.



A and C are the points of intersection of the line $y = x$ and the graph of f, and B is the minimum point of f.

(a) If A, B and C have x-coordinates $a\frac{\pi}{2}$, $b\frac{\pi}{6}$ and $c\frac{\pi}{2}$, where $a, b, c \in \mathbb{N}$, find the values of a, b and c.

(4)

(b) Find the range of f.

(3)

(c) Find the equation of the normal to the graph of f at the point C, giving your answer in the form $y = px + q$.

(5)

(d) The region S is rotated through 2π about the x-axis to generate a solid.

(i) Write down an integral that represents the volume V of this solid.

(ii) Show that $V = 6\pi^2$.

(7)
(Total 19 marks)

43. (a) Differentiate $f(x) = \arcsin x + 2\sqrt{1-x^2}$, $x \in [-1, 1]$.

(3)

(b) Find the coordinates of the point on the graph of $y = f(x)$ in $[-1, 1]$, where the gradient of the tangent to the curve is zero.

(3)
(Total 6 marks)

44. The acceleration in m s^{-2} of a particle moving in a straight line at time t seconds, $t \geq 0$, is given by the formula $a = -\frac{1}{2}v$. When $t = 0$, the velocity is 40 m s^{-1}.
Find an expression for v in terms of t.

(Total 6 marks)

45. The cubic curve $y = 8x^3 + bx^2 + cx + d$ has two distinct points P and Q, where the gradient is zero.

(a) Show that $b^2 > 24c$.

(4)

(b) Given that the coordinates of P and Q are $\left(\frac{1}{2}, -12\right)$ and $\left(-\frac{3}{2}, 20\right)$, respectively, find the values of b, c and d.

(4)
(Total 8 marks)

46. Using the substitution $x = 2 \sin \theta$, show that

$$\int \sqrt{4-x^2}\,dx = Ax\sqrt{4-x^2} + B\arcsin\left(\frac{x}{2}\right) + \text{constant},$$

where A and B are constants whose values you are required to find.

(Total 8 marks)

47. A helicopter H is moving vertically upwards with a speed of 10 m s^{-1}. The helicopter is h m directly above the point Q, which is situated on level ground. The helicopter is observed from the point P, which is also at ground level, and PQ = 40 m.
This information is represented in the diagram below.

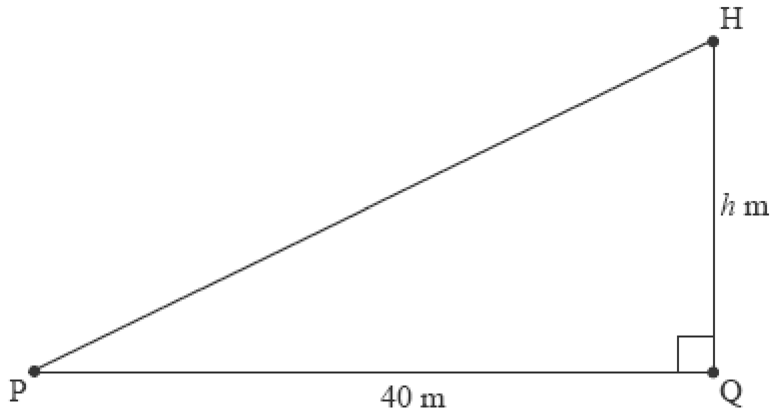


diagram not to scale

When $h = 30$,

(a) show that the rate of change of $\mathrm{H\hat{P}Q}$ is 0.16 radians per second;

(3)

(b) find the rate of change of PH.

(4)
(Total 7 marks)

48. Consider the graphs $y = e^{-x}$ and $y = e^{-x} \sin 4x$, for $0 \le x \le \frac{5\pi}{4}$.

(a) On the same set of axes draw, on graph paper, the graphs, for $0 \le x \le \frac{5\pi}{4}$.

Use a scale of 1 cm to $\frac{\pi}{8}$ on your x-axis and 5 cm to 1 unit on your y-axis.

(3)

(b) Show that the x-intercepts of the graph $y = e^{-x} \sin 4x$ are $\frac{n\pi}{4}$, $n = 0, 1, 2, 3, 4, 5$.

(3)

(c) Find the x-coordinates of the points at which the graph of $y = e^{-x} \sin 4x$ meets the graph of $y = e^{-x}$. Give your answers in terms of π.

(3)

(d) (i) Show that when the graph of $y = e^{-x} \sin 4x$ meets the graph of $y = e^{-x}$, their gradients are equal.

(ii) Hence explain why these three meeting points are not local maxima of the graph $y = e^{-x} \sin 4x$.

(6)

(e) (i) Determine the y-coordinates, y_1, y_2 and y_3, where $y_1 > y_2 > y_3$, of the local maxima of $y = e^{-x} \sin 4x$ for $0 \le x \le \frac{5\pi}{4}$. You do not need to show that they are maximum values, but the values should be simplified.

(ii) Show that y_1, y_2 and y_3 form a geometric sequence and determine the common ratio r.

(7)
(Total 22 marks)

49. (a) Calculate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}} dx$.

(6)

(b) Find $\int \tan^3 x \, dx$.

(3)
(Total 9 marks)

50. A certain population can be modelled by the differential equation $\frac{dy}{dt} = k\,y\cos kt$, where y is the population at time t hours and k is a positive constant.

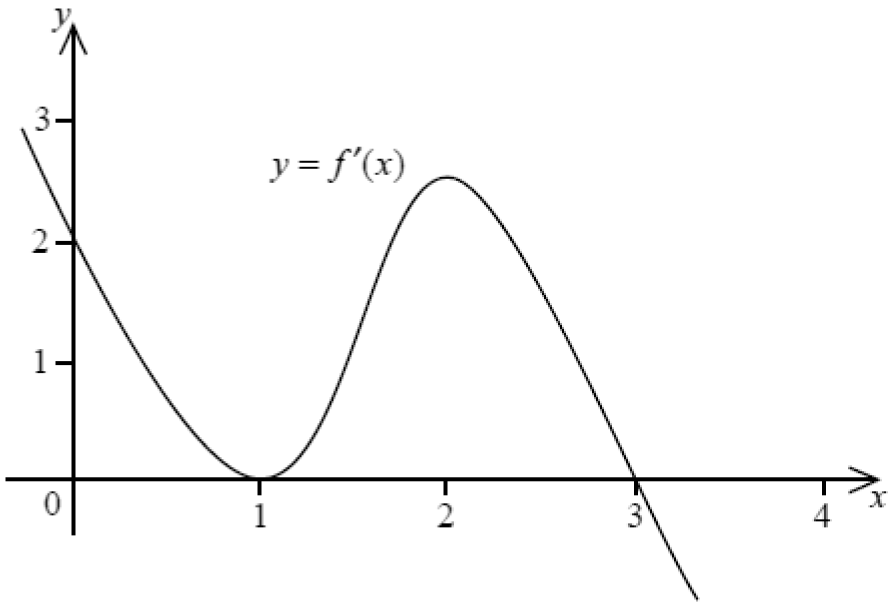
(a) Given that $y = y_0$ when $t = 0$, express y in terms of k, t and y_0.

(5)

(b) Find the ratio of the minimum size of the population to the maximum size of the population.

(2)
(Total 7 marks)

51. The diagram below shows a sketch of the gradient function $f'(x)$ of the curve $f(x)$.



On the graph below, sketch the curve $y = f(x)$ given that $f(0) = 0$. Clearly indicate on the graph any maximum, minimum or inflexion points.

(Total 5 marks)

52. A drinking glass is modelled by rotating the graph of $y = e^x$ about the y-axis, for $1 \le y \le 5$. Find the volume of the glass.

(Total 8 marks)

53. A tangent to the graph of $y = \ln x$ passes through the origin.

(a) Sketch the graphs of $y = \ln x$ and the tangent on the same set of axes, and hence find the equation of the tangent.

(11)

(b) Use your sketch to explain why $\ln x \le \frac{x}{e}$ for $x > 0$.

(1)

(c) Show that $x^e \le e^x$ for $x > 0$.

(3)

(d) Determine which is larger, π^e or e^π.

(2)
(Total 17 marks)

54. Consider the function f, defined by $f(x) = x - a\sqrt{x}$, where $x \ge 0$, $a \in \mathbb{R}^+$.

(a) Find in terms of a

(i) the zeros of f;

(ii) the values of x for which f is decreasing;

(iii) the values of x for which f is increasing;

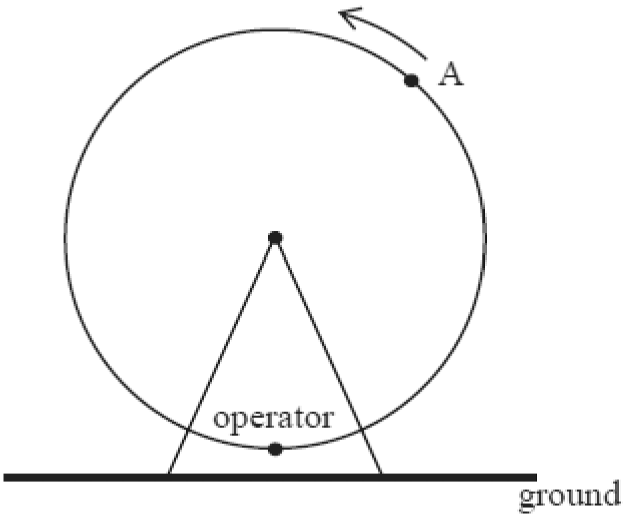
(iv) the range of f.

(10)

(b) State the concavity of the graph of *f*.

(1)
(Total 11 marks)

55. Below is a sketch of a Ferris wheel, an amusement park device carrying passengers around the rim of the wheel.



(a) The circular Ferris wheel has a radius of 10 metres and is revolving at a rate of 3 radians per minute. Determine how fast a passenger on the wheel is going vertically upwards when the passenger is at point A, 6 metres higher than the centre of the wheel, and is rising.

(7)

(b) The operator of the Ferris wheel stands directly below the centre such that the bottom of the Ferris wheel is level with his eyeline. As he watches the passenger his line of sight makes an angle α with the horizontal. Find the rate of change of α at point A.

(3)
(Total 10 marks)

56. If $f(x) = x - 3x^{\frac{2}{3}}$, $x > 0$,

(a) find the x-coordinate of the point P where $f'(x) = 0$;

(2)

(b) determine whether P is a maximum or minimum point.

(3)
(Total 5 marks)

57. Find the area between the curves $y = 2 + x - x^2$ and $y = 2 - 3x + x^2$.

(Total 7 marks)

58. The region bounded by the curve $y = \dfrac{\ln(x)}{x}$ and the lines $x = 1$, $x = \mathrm{e}$, $y = 0$ is rotated through 2π radians about the x-axis.

Find the volume of the solid generated.

(Total 6 marks)

59. The function f is defined by $f(x) = x\,\mathrm{e}^{2x}$.

It can be shown that $f^{(n)}(x) = (2^n x + n\,2^{n-1})\,\mathrm{e}^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

(a) By considering $f^{(n)}(x)$ for $n = 1$ and $n = 2$, show that there is one minimum point P on the graph of f, and find the coordinates of P.

(7)

(b) Show that f has a point of inflexion Q at $x = -1$.

(5)

(c) Determine the intervals on the domain of f where f is

(i) concave up;

(ii) concave down.

(2)

(d) Sketch f, clearly showing any intercepts, asymptotes and the points P and Q.

(4)

(e) Use mathematical induction to prove that $f^{(n)}(x) = (2^n x + n2^{n-1})\,e^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

(9)
(Total 27 marks)

60. A gourmet chef is renowned for her spherical shaped soufflé. Once it is put in the oven, its volume increases at a rate proportional to its radius.

(a) Show that the radius r cm of the soufflé, at time t minutes after it has been put in the oven, satisfies the differential equation $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r}$, where k is a constant.

(5)

(b) Given that the radius of the soufflé is 8 cm when it goes in the oven, and 12 cm when it's cooked 30 minutes later, find, to the nearest cm, its radius after 15 minutes in the oven.

(8)
(Total 13 marks)

61. Consider the curve with equation $x^2 + xy + y^2 = 3$.

(a) Find in terms of k, the gradient of the curve at the point $(-1, k)$.

(5)

(b) Given that the tangent to the curve is parallel to the x-axis at this point, find the value of k.

(1)
(Total 6 marks)

62. Show that $\int_0^{\frac{\pi}{6}} x \sin 2x \,\mathrm{d}x = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$.

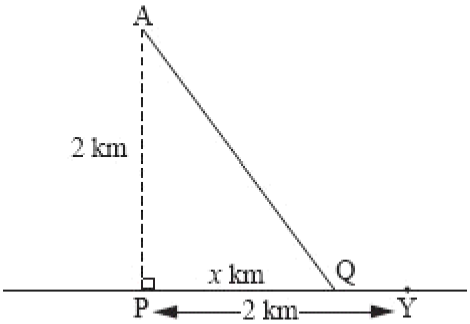
(Total 6 marks)

63. A normal to the graph of $y = \arctan(x - 1)$, for $x > 0$, has equation $y = -2x + c$, where $c \in \mathbb{R}$.

Find the value of c.

(Total 6 marks)

64. André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that AP = 2 km and PY = 2 km. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.



When André swims he covers 1 km in $5\sqrt{5}$ minutes. When he runs he covers 1 km in 5 minutes.

(a) If PQ = x km, $0 \le x \le 2$, find an expression for the time T minutes taken by André to reach point Y.

(4)

(b) Show that $\dfrac{dT}{dx} = \dfrac{5\sqrt{5}\,x}{\sqrt{x^2+4}} - 5$.

(3)

(c) (i) Solve $\dfrac{dT}{dx} = 0$.

(ii) Use the value of x found in **part (c) (i)** to determine the time, T minutes, taken for André to reach point Y.

(iii) Show that $\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = \frac{20\sqrt{5}}{\left(x^2+4\right)^{\frac{3}{2}}}$ and **hence** show that the time found in **part (c) (ii)** is a minimum.

(11)
(Total 18 marks)

65. Find the gradient of the tangent to the curve $x^3 y^2 = \cos(\pi y)$ at the point $(-1, 1)$.

(Total 6 marks)

66. By using an appropriate substitution find

$$\int \frac{\tan(\ln y)}{y}\,\mathrm{d}y,\ y > 0.$$

(Total 6 marks)

67. A family of cubic functions is defined as $f_k(x) = k^2x^3 - kx^2 + x$, $k \in \mathbb{Z}^+$.

(a) Express in terms of k

(i) $f'_k(x)$ and $f''_k(x)$;

(ii) the coordinates of the points of inflexion P_k on the graphs of f_k.

(6)

(b) Show that all P_k lie on a straight line and state its equation.

(2)

(c) Show that for all values of k, the tangents to the graphs of f_k at P_k are parallel, and find the equation of the tangent lines.

(5)
(Total 13 marks)

68. The curve $y = e^{-x} - x + 1$ intersects the x-axis at P.

(a) Find the x-coordinate of P.

(2)

(b) Find the area of the region completely enclosed by the curve and the coordinate axes.

(3)
(Total 5 marks)

69. Consider the curve with equation $f(x) = e^{-2x^2}$ for $x < 0$.

Find the coordinates of the point of inflexion and justify that it is a point of inflexion.

(Total 7 marks)

70. A particle moves in a straight line in a positive direction from a fixed point O.

The velocity v m s^{-1}, at time t seconds, where $t \geq 0$, satisfies the differential equation

$$\frac{dv}{dt} = \frac{-v(1+v^2)}{50}.$$

The particle starts from O with an initial velocity of 10 m s^{-1}.

(a) (i) Express as a definite integral, the time taken for the particle's velocity to decrease from 10 m s^{-1} to 5 m s^{-1}.

(ii) **Hence** calculate the time taken for the particle's velocity to decrease from 10 m s^{-1} to 5 m s^{-1}.

(5)

(b) (i) Show that, when $v > 0$, the motion of this particle can also be described by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-(1+v^2)}{50}$ where x metres is the displacement from O.

(ii) Given that $v = 10$ when $x = 0$, solve the differential equation expressing x in terms of v.

(iii) **Hence** show that $v = \dfrac{10 - \tan\frac{x}{50}}{1 + 10\tan\frac{x}{50}}$.

(14)
(Total 19 marks)

71. (a) Using l'Hopital's Rule, show that $\lim_{x\to\infty} x\mathrm{e}^{-x} = 0$.

(2)

(b) Determine $\int_0^a x\mathrm{e}^{-x}\,\mathrm{d}x$.

(5)

(c) Show that the integral $\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x$ is convergent and find its value.

(2)
(Total 9 marks)

72. Calculate the exact value of $\int_1^{\mathrm{e}} x^2 \ln x\,\mathrm{d}x$.

(Total 5 marks)

73. Find the equation of the normal to the curve $5xy^2 - 2x^2 = 18$ at the point (1, 2).

(Total 7 marks)

74. (a) Use the derivatives of sin x and cos x to show that the derivative of tan x is $\sec^2 x$.

(3)

(b) Hence by using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, show that the derivative of arctan x is $\frac{1}{1+x^2}$.

(4)
(Total 7 marks)

75. A packaging company makes boxes for chocolates. An example of a box is shown below. This box is closed and the top and bottom of the box are identical regular hexagons of side x cm.

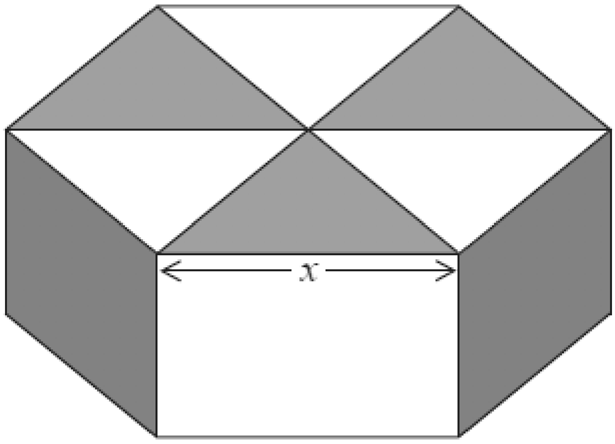


diagram not to scale

(a) Show that the area of each hexagon is $\frac{3\sqrt{3}x^2}{2}$ cm^2.

(1)

(b) Given that the volume of the box is 90 cm^3, show that when $x = \sqrt[3]{20}$ the total surface area of the box is a minimum, justifying that this value gives a minimum.

(7)
(Total 8 marks)

76. If $y = \ln\left(\frac{1}{3}(1+e^{-2x})\right)$, show that $\frac{dy}{dx} = \frac{2}{3}(e^{-y} - 3)$.

(Total 7 marks)

77. The population of mosquitoes in a specific area around a lake is controlled by pesticide. The rate of decrease of the number of mosquitoes is proportional to the number of mosquitoes at any time t. Given that the population decreases from 500 000 to 400 000 in a five year period, find the time it takes in years for the population of mosquitoes to decrease by half.

(Total 8 marks)

78. The function f is defined by $f(x) = x\sqrt{9-x^2} + 2\arcsin\left(\frac{x}{3}\right)$.

(a) Write down the largest possible domain, for each of the two terms of the function, f, and hence state the largest possible domain, D, for f.

(2)

(b) Find the volume generated when the region bounded by the curve $y = f(x)$, the x-axis, the y-axis and the line $x = 2.8$ is rotated through 2π radians about the x-axis.

(3)

(c) Find $f'(x)$ in simplified form.

(5)

(d) **Hence** show that $\int_{-p}^{p} \frac{11-2x^2}{\sqrt{9-x^2}}dx = 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$, where $p \in D$.

(2)

(e) Find the value of p that maximises the value of the integral in (d).

(2)

(f) (i) Show that $f''(x) = \frac{x(2x^2-25)}{(9-x^2)^{\frac{3}{2}}}$.

(ii) Hence justify that $f(x)$ has a point of inflexion at $x = 0$, but not at $x = \pm\sqrt{\frac{25}{2}}$.

(7)
(Total 21 marks)

79. (a) Show that the solution of the differential equation

$$\frac{dy}{dx} = \cos x \cos^2 y,$$

given that $y = \frac{\pi}{4}$ when $x = \pi$, is $y = \arctan(1 + \sin x)$.

(5)

(b) Determine the value of the constant a for which the following limit exists

$$\lim_{x \to \frac{\pi}{2}} \frac{\arctan(1 + \sin x) - a}{\left(x - \frac{\pi}{2}\right)^2}$$

and evaluate that limit.

(12)
(Total 17 marks)

80. The function f' is given by $f'(x) = 2\sin\left(5x - \frac{\pi}{2}\right)$.

(a) Write down $f''(x)$.

(2)

(b) Given that $f\left(\frac{\pi}{2}\right) = 1$, find $f(x)$.

(4)
(Total 6 marks)

81. Find the gradient of the normal to the curve $3x^2y + 2xy^2 = 2$ at the point (1, –2).

(Total 6 marks)

82. Solve the differential equation $x\frac{dy}{dx} - y^2 = 1$, given that $y = 0$ when $x = 2$.
Give your answer in the form $y = f(x)$.

(Total 6 marks)

83. (a) Sketch the curves $y = x^2$ and $y = |x|$.

(3)

(b) Find the sum of the areas of the regions enclosed by the curves $y = x^2$ and $y = |x|$.

(4)
(Total 7 marks)

84. The acceleration of a body is given in terms of the displacement s metres as
$a = \frac{2s}{s^2 + 1}$.

(a) Give a formula for the velocity as a function of the displacement given that when $s = 1$ metre, $v = 2$ m s^{-1}.

(7)

(b) Hence find the velocity when the body has travelled 5 metres.

(2)
(Total 9 marks)

85. A curve C is defined implicitly by $xe^y = x^2 + y^2$. Find the equation of the tangent to C at the point (1, 0).

(Total 7 marks)

86. The function f is defined by $f(x) = (\ln(x-2))^2$. Find the coordinates of the point of inflexion of f.

(Total 9 marks)

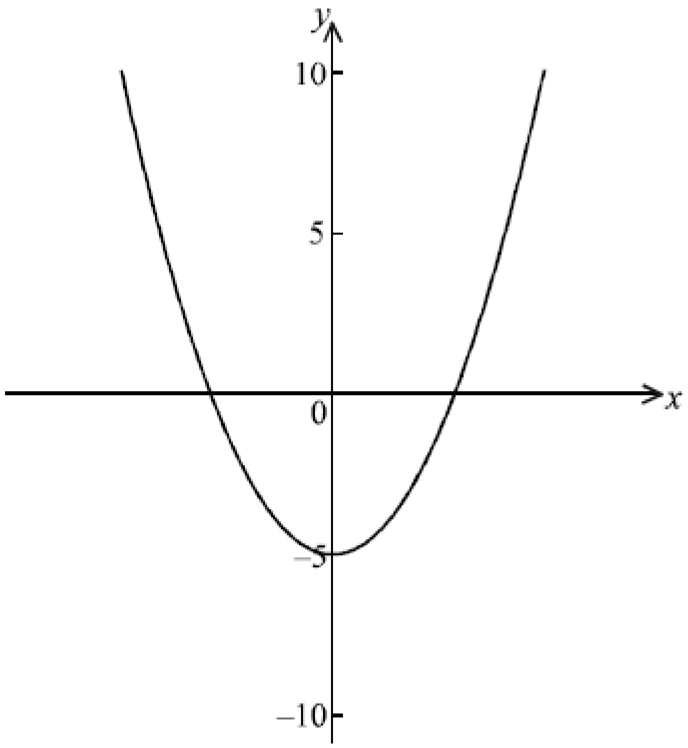
87. Find $\int_1^{e} \frac{(\ln x)^3}{x}\,dx$.

(Total 5 marks)

88. Find $\int_1^{\sqrt{3}} \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$.

(Total 11 marks)

89. The curve $y = x^2 - 5$ is shown below.



A point P on the curve has x-coordinate equal to a.

(a) Show that the distance OP is $\sqrt{a^4 - 9a^2 + 25}$.

(2)

(b) Find the values of a for which the curve is closest to the origin.

(5)
(Total 7 marks)

90. Find $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\sqrt{\cos x}} \mathrm{d}x$.

(Total 7 marks)

91. Use the substitution $u = x + 2$ to find $\int \frac{x^3}{(x+2)^2}\mathrm{d}x$.

(Total 6 marks)

92. It is given that

$$f(x) = \frac{18(x-1)}{x^2},\ f'(x) = \frac{18(2-x)}{x^3}, \text{ and } f''(x) = \frac{36(x-3)}{x^4},\ x \in \mathbb{R},\ x \neq 0.$$

(a) Find

(i) the zero(s) of $f(x)$;

(ii) the equations of the asymptotes;

(iii) the coordinates of the local maximum and justify it is a maximum;

(iv) the interval(s) where $f(x)$ is concave up.

(9)

(b) Hence sketch the graph of $y = f(x)$.

(5)
(Total 14 marks)

93. The function f is defined on the domain $x \geq 1$ by $f(x) = \frac{\ln x}{x}$.

(a) (i) Show, by considering the first and second derivatives of f, that there is one maximum point on the graph of f.

(ii) State the **exact** coordinates of this point.

(iii) The graph of f has a point of inflexion at P. Find the x-coordinate of P.

(12)

Let R be the region enclosed by the graph of f, the x-axis and the line $x = 5$.

(b) Find the **exact** value of the area of R.

(6)
(Total 18 marks)

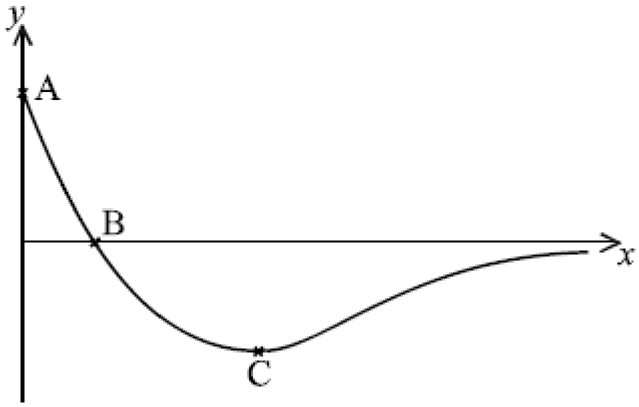
94. (a) Find the root of the equation $e^{2-2x} = 2e^{-x}$ giving the answer as a logarithm.

(4)

(b) The curve $y = e^{2-2x} - 2e^{-x}$ has a minimum point. Find the coordinates of this minimum.

(7)

(c) The curve $y = e^{2-2x} - 2e^{-x}$ is shown below.



Write down the coordinates of the points A, B and C.

(3)

(d) Hence state the set of values of k for which the equation $e^{2-2x} - 2e^{-x} = k$ has two distinct positive roots.

(2)
(Total 16 marks)

95. The function f is defined on the domain $x \geq 0$ by $f(x) = \frac{x^2}{e^x}$.

(a) Find the maximum value of $f(x)$, and justify that it is a maximum.

(10)

(b) Find the x coordinates of the points of inflexion on the graph of f.

(3)

(c) Evaluate $\int_0^1 f(x)\mathrm{d}x$.

(8)
(Total 21 marks)

96. Find the value of the integral $\int_0^4 \left|x^2 - 4\right| \mathrm{d}x$.

(Total 7 marks)

97. Find the gradient of the curve $e^{xy} + \ln(y^2) + e^y = 1 + e$ at the point (0, 1).

(Total 7 marks)