MATH HL2 – CORE TOPICS – CIRCULAR FUNCTIONS & TRIG (SOLUTIONS)

1. (a) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{2\sin\theta\cos\theta}{1+2\cos^2\theta - 1}$ M1

Note: Award M1 for use of double angle formulae.

$= \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta}$ A1

$= \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta$ AG

(b) $\tan\dfrac{\pi}{8} = \dfrac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$ (M1)

$\cot\dfrac{\pi}{8} = \dfrac{1+\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}$ M1

$= \dfrac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

$= 1 + \sqrt{2}$ A1

[5]

2. (a) area of AOP = $\dfrac{1}{2}r^2\sin\theta$ A1

(b) TP = $r\tan\theta$ (M1)

area of POT = $\dfrac{1}{2}r(r\tan\theta)$

$= \dfrac{1}{2}r^2\tan\theta$ A1

(c) area of sector OAP = $\dfrac{1}{2}r^2\theta$ A1

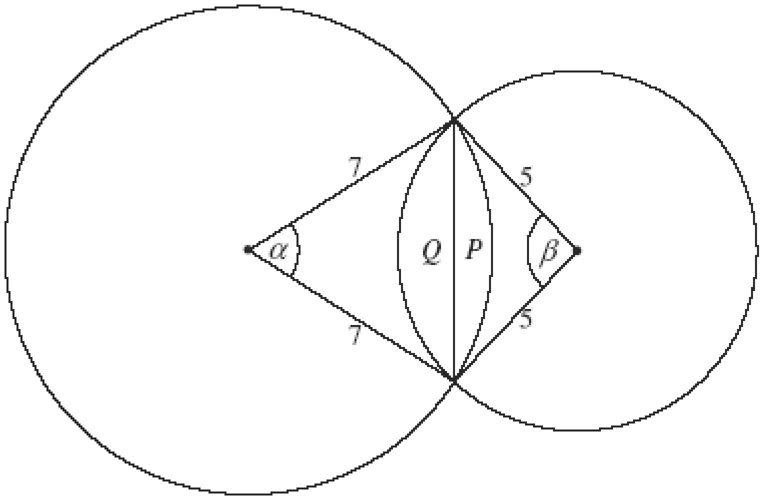
area of triangle OAP < area of sector OAP < area of triangle POT R1

$\dfrac{1}{2}r^2\sin\theta < \dfrac{1}{2}r^2\theta < \dfrac{1}{2}r^2\tan\theta$

$\sin\theta < \theta < \tan\theta$ AG

[5]

3.



$\alpha = 2 \arcsin\left(\frac{4.5}{7}\right)$ ($\Rightarrow$ $\alpha = 1.396... = 80.010°$ ...) M1(A1)

$\beta = 2 \arcsin\left(\frac{4.5}{5}\right)$ ($\Rightarrow$ $\beta = 2.239... = 128.31°$...) (A1)

Note: Allow use of cosine rule.

area $P = \frac{1}{2} \times 7^2 \times (\alpha - \sin \alpha) = 10.08...$ M1(A1)

area $Q = \frac{1}{2} \times 5^2 \times (\beta - \sin \beta) = 18.18...$ (A1)

Note: The M1 is for an attempt at area of sector minus area of triangle.

Note: The use of degrees correctly converted is acceptable.

area = 28.3(cm^2) A1

[7]

4. area of triangle POQ = $\frac{1}{2}8^2 \sin 59°$ M1

$= 27.43$ (A1)

area of sector = $\pi 8^2 \frac{59}{360}$ M1

$= 32.95$ (A1)

area between arc and chord = 32.95 – 27.43

$= 5.52$ (cm^2) A1

[5]

5. let the length of one side of the triangle be x
consider the triangle consisting of a side of the triangle and two radii

EITHER

$x^2 = r^2 + r^2 - 2r^2 \cos 120°$ M1
$= 3r^2$

OR

$x = 2r \cos 30°$ M1

THEN

$x = r\sqrt{3}$ A1
so perimeter = $3\sqrt{3}\,r$ A1
now consider the area of the triangle
area = $3 \times \frac{1}{2} r^2 \sin 120°$ M1
$= 3 \times \frac{\sqrt{3}}{4} r^2$ A1

$$\frac{P}{A} = \frac{3\sqrt{3}r}{\frac{3\sqrt{3}}{4} r^2}$$

$= \frac{4}{r}$ A1

Note: Accept alternative methods

[6]

6. (a) $u = \frac{1}{x} \Rightarrow \mathrm{d}u = -\frac{1}{x^2}\,\mathrm{d}x$ M1

$\Rightarrow \mathrm{d}x = -\frac{\mathrm{d}u}{u^2}$ A1

$\int_1^{\alpha} \frac{1}{1+x^2}\mathrm{d}x = -\int_1^{\frac{1}{\alpha}} \frac{1}{1+\left(\frac{1}{u}\right)^2} \frac{\mathrm{d}u}{u^2}$ A1M1A1

Note: Award A1 for correct integrand and M1A1 for correct limits.

$= \int_{\frac{1}{\alpha}}^{1} \frac{1}{1+u^2}\mathrm{d}u$ (upon interchanging the two limits) AG

(b) $\arctan x_1^{\alpha} = \arctan u_{\frac{1}{\alpha}}^{1}$ A1

$\arctan \alpha - \frac{\pi}{4} = \frac{\pi}{4} - \arctan \frac{1}{\alpha}$ A1

$\arctan \alpha$ or $+ \arctan \frac{1}{\alpha} = \frac{\pi}{2}$ AG

[7]

7. (a) $\sin (2n + 1)x \cos x - \cos (2n + 1)x \sin x = \sin (2n + 1)x - x$ M1A1

$= \sin 2nx$ AG

(b) if $n = 1$ M1

LHS = $\cos x$

RHS = $\frac{\sin 2x}{2\sin x} = \frac{2\sin x \cos x}{2\sin x} = \cos x$ M1

so LHS = RHS and the statement is true for $n = 1$ R1

assume true for $n = k$ M1

Note: Only award M1 if the word true appears.
Do not award M1 for 'let $n = k$' only.
Subsequent marks are independent of this M1.

so $\cos x + \cos 3x + \cos 5x + ... + \cos(2k - 1)x = \frac{\sin 2kx}{2\sin x}$

if $n = k + 1$ then

$\cos x + \cos 3x + \cos 5x + ... + \cos(2k - 1)x + \cos(2k + 1)x$ M1

$= \frac{\sin 2kx}{2\sin x} \cos (2k + 1)x$ A1

$= \frac{\sin 2kx + 2\cos(2k+1)x \sin x}{2\sin x}$ M1

$= \frac{\sin(2k+1)x \cos x - \cos(2k+1)x \sin x + 2\cos(2k+1)x \sin x}{2\sin x}$ M1

$= \frac{\sin(2k+1)x \cos x + \cos(2k+1)x \sin x}{2\sin x}$ A1

$= \frac{\sin(2k+2)x}{2\sin x}$ M1

$= \frac{\sin 2(k+1)x}{2\sin x}$ A1

so if true for $n = k$, then also true for $n = k + 1$

as true for $n = 1$ then true for all $n \in \mathbb{Z}^+$ R1

Note: Final R1 is independent of previous work.

(c) $\dfrac{\sin 4x}{2\sin x} = \dfrac{1}{2}$ M1A1

$\sin 4x = \sin x$

$4x = x \Rightarrow x = 0$ but this is impossible

$4x = \pi - x \Rightarrow x = \dfrac{\pi}{5}$ A1

$4x = 2\pi + x \Rightarrow x = \dfrac{2\pi}{3}$ A1

$4x = 3\pi - x \Rightarrow x = \dfrac{3\pi}{5}$ A1

for not including any answers outside the domain R1

Note: Award the first M1A1 for correctly obtaining $8\cos^3 x - 4\cos x - 1 = 0$ or equivalent and subsequent marks as appropriate including the answers $\arccos\left(-\dfrac{1}{2}, \dfrac{1 \pm \sqrt{5}}{4}\right)$.

[20]

8. (a) the differential equation is separable and can be written as (M1)

$\int -y^{-2}\mathrm{d}y = \int \cos^2 x\mathrm{d}x$ (or equivalent) A1

$= \int \dfrac{1+\cos 2x}{2}\mathrm{d}x$ A1

$\dfrac{1}{y} = \dfrac{1}{2}x + \dfrac{1}{4}\sin 2x (+C)$ A1A1

when $x = 0$, $y = 1$ M1

$C = 1$

$y = \dfrac{1}{\frac{1}{2}x + \frac{1}{4}\sin 2x + 1}$ A1

(b) (i) recognizing use of $(1 + \tan x)^2$ (M1)

$(1 + \tan x)^2 = 1 + 2\tan x + \tan^2 x \geq 1 + \tan^2 x = \sec^2 x$ A1

(since all terms are positive)

$(1 + \tan x)^2 \geq \sec^2 x$

$\sec^2 x = 1 + \tan^2 x \geq 1$ A1

$\Rightarrow (1 + \tan x)^2 \geq \sec^2 x \geq 1$

since all terms are positive, taking square root gives R1

$1 \leq \sec x \leq 1 + \tan x$ AG

(ii) $\int_0^{\frac{\pi}{4}} \mathrm{d}x \le \int_0^{\frac{\pi}{4}} \sec x \mathrm{d}x \le \int_0^{\frac{\pi}{4}} 1 + \tan x \mathrm{d}x$ M1

$x_0^{\frac{\pi}{4}} \le \int_0^{\frac{\pi}{4}} \sec x \mathrm{d}x \le x - \ln \cos x_0^{\frac{\pi}{4}}$ M1A1

$\frac{\pi}{4} \le \int_0^{\frac{\pi}{4}} \sec x \mathrm{d}x \le \frac{\pi}{4} - \ln \frac{1}{\sqrt{2}}$ A1

$\frac{\pi}{4} \le \int_0^{\frac{\pi}{4}} \sec x \mathrm{d}x \le \frac{\pi}{4} + \frac{1}{2} \ln 2$ AG

[15]

9. $a = 3$ A1
$c = 2$ A1
period = $\frac{2\pi}{b} = 3$ (M1)
$b = \frac{2\pi}{3}$ (= 2.09) A1

[4]

10. AC = AB = 10 (cm) A1
triangle OBC is equilateral (M1)
BC = 6 (cm) A1

EITHER

$\mathrm{B\hat{A}C} = 2 \arcsin \frac{3}{10}$ M1A1

$\mathrm{B\hat{A}C}$ = 34.9° (accept 0.609 radians) A1

OR

$\cos \mathrm{B\hat{A}C} = \frac{10^2 + 10^2 - 6^2}{2 \times 10 \times 10} = \frac{164}{200}$ M1A1

$\mathrm{B\hat{A}C}$ = 34.9° (accept 0.609 radians) A1

Note: Other valid methods may be seen.

[6]

11. use of cosine rule: BC = $\sqrt{(8^2 + 7^2 - 2\times 7\times 8\cos 70)}$ = 8.6426... (M1)A1

Note: Accept an expression for BC^2.

BD = 5.7617... (CD = 2.88085...) A1

use of sine rule: $\hat{B} = \arcsin\left(\dfrac{7\sin 70}{BC}\right)$ = 49.561...° ($\hat{C}$ = 60.4387...°) (M1)A1

use of cosine rule: AD = $\sqrt{8^2 + BD^2 - 2\times BD\times 8\cos B}$ = 6.12 (cm) A1

Note: Scale drawing method not acceptable.

[6]

12. (a) the area of the first sector is $\frac{1}{2}2^2\theta$ (A1)

the sequence of areas is 2θ, $2k\theta$, $2k^2\theta$... (A1)

the sum of these areas is $2\theta(1 + k + k^2 + ...)$ (M1)

$= \dfrac{2\theta}{1-k} = 4\pi$ M1A1

hence $\theta = 2\pi(1 - k)$ AG

Note: Accept solutions where candidates deal with angles instead of area.

(b) the perimeter of the first sector is $4 + 2\theta$ (A1)

the perimeter of the third sector is $4 + 2k^2\theta$ (A1)

the given condition is $4 + 2k^2\theta = 2 + \theta$ M1

which simplifies to $2 = \theta(1 - 2k^2)$ A1

eliminating θ, obtain cubic in k: $\pi(1 - k)(1 - 2k^2) - 1 = 0$ or equivalent A1

solve for $k = 0.456$ and then $\theta = 3.42$ A1A1

[12]

13. (a) $\dfrac{\pi}{4} - \arccos x \geq 0$

$\arccos x \leq \dfrac{\pi}{4}$ (M1)

$x \geq \dfrac{\sqrt{2}}{2}$ $\left(\text{accept } x \geq \dfrac{1}{\sqrt{2}}\right)$ (A1)

since $-1 \leq x \leq 1$ (M1)

$\Rightarrow \dfrac{\sqrt{2}}{2} \leq x \leq 1$ $\left(\text{accept } \dfrac{1}{\sqrt{2}} \leq x \leq 1\right)$ A1

Note: Penalize the use of < instead of ≤ only once.

(b) $y = \sqrt{\frac{\pi}{4} - \arccos x} \Rightarrow x = \cos\left(\frac{\pi}{4} - y^2\right)$ M1A1

$f^{-1} : x \to \cos\left(\frac{\pi}{4} - x^2\right)$ A1

$0 \le x \le \sqrt{\frac{\pi}{4}}$ A1

[8]

14. METHOD 1

(a) $|\boldsymbol{a} - \boldsymbol{b}| = \sqrt{|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\alpha}$ M1

$= \sqrt{2 - 2\cos\alpha}$ A1

$|\boldsymbol{a} + \boldsymbol{b}| = \sqrt{|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos(\pi - \alpha)}$

$= \sqrt{2 + 2\cos\alpha}$ A1

Note: Accept the use of a, b for $|\boldsymbol{a}|$, $|\boldsymbol{b}|$.

(b) $\sqrt{2 + 2\cos\alpha} = 3\sqrt{2 - 2\cos\alpha}$ M1

$\cos\alpha = \frac{4}{5}$ A1

METHOD 2

(a) $|\boldsymbol{a} - \boldsymbol{b}| = 2\sin\frac{\alpha}{2}$ M1A1

$|\boldsymbol{a} + \boldsymbol{b}| = 2\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = 2\cos\frac{\alpha}{2}$ A1

Note: Accept the use of a, b for $|\boldsymbol{a}|$, $|\boldsymbol{b}|$.

(b) $2\cos\frac{\alpha}{2} = 6\sin\frac{\alpha}{2}$

$\tan\frac{\alpha}{2} = \frac{1}{3} \Rightarrow \cos^2\frac{\alpha}{2} = \frac{9}{10}$ M1

$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = \frac{4}{5}$ A1

[5]

15. (a) (i) the period is 2 A1

(ii) $v = \frac{\mathrm{d}s}{\mathrm{d}t} = 2\pi\cos(\pi t) + 2\pi\cos(2\pi t)$ (M1)A1

$a = \frac{dv}{dt} = -2\pi^2 \sin(\pi t) - 4\pi^2 \sin(2\pi t)$ (M1)A1

(iii) $v = 0$

$2\pi(\cos(\pi t) + \cos(2\pi t)) = 0$

EITHER

$\cos(\pi t) + 2\cos^2(\pi t) - 1 = 0$ M1

$(2\cos(\pi t) - 1)(\cos(\pi t) + 1) = 0$ (A1)

$\cos(\pi t) = \frac{1}{2}$ or $\cos(\pi t) = -1$ A1

$t = \frac{1}{3}$, $t = 1$ A1

$t = \frac{5}{3}, t = \frac{7}{3}, t = \frac{11}{3}$ $t = 3$ A1

OR

$2\cos\left(\frac{\pi t}{2}\right)\cos\left(\frac{3\pi t}{2}\right) = 0$ M1

$\cos\left(\frac{\pi t}{2}\right) = 0$ or $\cos\left(\frac{3\pi t}{2}\right) = 0$ A1A1

$t = \frac{1}{3}, 1$ A1

$t = \frac{5}{3}, \frac{7}{3}, 3, \frac{11}{3}$ A1

(b) $P(n)$: $f^{(2n)}(x) = (-1)^n(Aa^{2n} \sin(ax) + Bb^2n \sin(bx))$

$P(1)$: $f''(x) = (Aa\cos(ax) + Bb\cos(bx))'$ M1

$= -Aa^2 \sin(ax) - Bb^2 \sin(bx)$

$= -1(Aa^2 \sin(ax) + Bb^2 \sin(bx))$ A1

$\therefore P(1)$ true

assume that

$P(k)$: $f^{(2k)}(x) = (-1)^k [(Aa^{2k} \sin(ax) + Bb^{2k} \sin(bx))$ is true M1

consider $P(k + 1)$

$f^{(2k+1)}(x) = (-1)^k (Aa^{2k+1} \cos(ax) + Bb^{2k+1} \cos(bx))$ M1A1

$f^{(2k+2)}(x) = (-1)^k (-Aa^{2k+2} \sin(ax) + Bb^{2k+2} \sin(bx))$ A1

$= (-1)^{k+1} (Aa^{2k+2} \sin(ax) + Bb^{2k+2} \sin(bx))$ A1

$P(k)$ true implies $P(k + 1)$ true, $P(1)$ true so $P(n)$ true $\forall n \in \mathbb{Z}^+$ R1

Note: Award the final R1 only if the previous three M marks have been awarded.

[18]

16. (a) area $= \frac{1}{2} \times BC \times AB \times \sin B$ (M1)

$\left(10 = \frac{1}{2} \times 5 \times 6 \times \sin B\right)$

$\sin \hat{B} = \frac{2}{3}$ A1

(b) $\cos B = \pm\frac{\sqrt{5}}{3}$ ($= \pm 0.7453...$) or $B = 41.8...$ and $138.1...$ (A1)

$AC^2 = BC^2 + AB^2 - 2 \times BC \times AB \times \cos B$ (M1)

$AC = \sqrt{5^2 + 6^2 - 2\times5\times6\times0.7453...}$ or $\sqrt{5^2 + 6^2 + 2\times5\times6\times0.7453...}$

AC = 4.03 or 10.28 A1A1

[6]

17. (a) $-2 = 1 + k\sin\left(\frac{\pi}{6}\right)$ M1

$-3 = \frac{1}{2}k$ A1

$k = -6$ AG N0

(b) **METHOD 1**

maximum $\Rightarrow$ $\sin x = -1$ M1

$a = \frac{3\pi}{2}$ A1

$b = 1 - 6(-1)$

$= 7$ A1 N2

METHOD 2

$y' = 0$ M1

$k\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, ...$

$a = \frac{3\pi}{2}$ A1

$b = 1 - 6(-1)$

$= 7$ A1 N2

Note: Award A1A1 for $(\frac{3\pi}{2}, 7)$

[5]

18. (a) **METHOD 1**

let $x = \arctan\frac{1}{2} \Rightarrow \tan x = \frac{1}{2}$ and $y = \arctan\frac{1}{3} \Rightarrow \tan y = \frac{1}{3}$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1$ M1

so, $x + y = \arctan 1 = \dfrac{\pi}{4}$ A1AG

METHOD 2

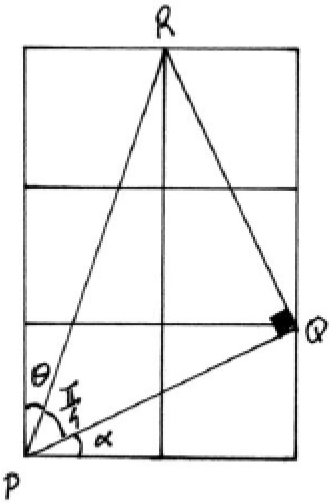
for $x, y > 0$, $\arctan x + \arctan y = \arctan\left(\dfrac{x+y}{1-xy}\right)$ if $xy < 1$ M1

so, $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right) = \dfrac{\pi}{4}$ A1AG

METHOD 3

an appropriate sketch M1

e.g.



correct reasoning leading to $\dfrac{\pi}{4}$ R1AG

(b) **METHOD 1**

$\arctan(2) + \arctan(3) = \frac{\pi}{2} - \arctan\left(\frac{1}{2}\right) + \frac{\pi}{2} - \arctan\left(\frac{1}{3}\right)$ (M1)

$= \pi - \left(\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right)$ (A1)

Note: Only one of the previous two marks may be implied.

$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ A1 N1

METHOD 2

let $x = \arctan 2 \Rightarrow \tan x = 2$ and $y = \arctan 3 \Rightarrow \tan y = 3$

$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{2+3}{1-2\times 3} = -1$ (M1)

as $\frac{\pi}{4} < x < \frac{\pi}{2}$ $\left(\text{accept } 0 < x < \frac{\pi}{2}\right)$

and $\frac{\pi}{4} < y < \frac{\pi}{2}$ $\left(\text{accept } 0 < y < \frac{\pi}{2}\right)$

$\frac{\pi}{2} < x + y < \pi$ (accept $0 < x + y < \pi$) (R1)

Note: Only one of the previous two marks may be implied.

so, $x + y = \frac{3\pi}{4}$ A1 N1

METHOD 3

for $x, y > 0$, $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi$ if $xy > 1$ (M1)

so, $\arctan 2 + \arctan 3 = \arctan\left(\frac{2+3}{1-2\times 3}\right) + \pi$ (A1)

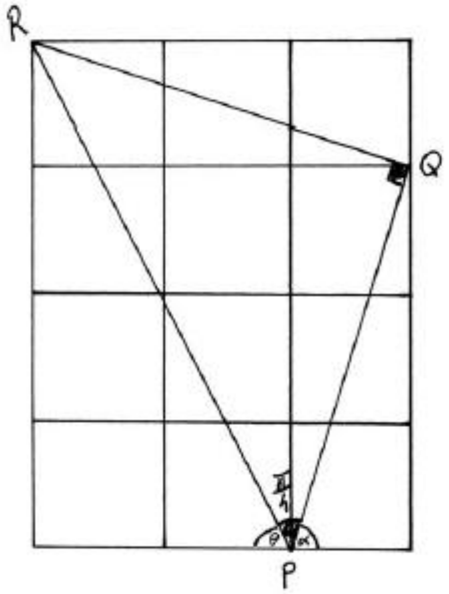
Note: Only one of the previous two marks may be implied.

$= \frac{3\pi}{4}$ A1 N1

METHOD 4

an appropriate sketch M1

e.g.



correct reasoning leading to $\frac{3\pi}{4}$ R1A1

[5]

19. $A = \frac{\theta}{2}(R^2 - r^2)$ A1

$B = \frac{\theta}{2}r^2$ A1

from $A : B = 2:1$, we have $R^2 - r^2 = 2r^2$ M1

$R = \sqrt{3}r$ (A1)

hence exact value of the ratio $R : r$ is $\sqrt{3}\ :1$ A1 N0

[5]

20. (a) a reasonable attempt to show either that $n^2 + n + 1 > 2n + 1$ or $n^2 + n + 1 > n^2 - 1$ M1

complete solution to each inequality A1A1

(b) $\cos\theta = \dfrac{(2n+1)^2 + (n^2-1)^2 - (n^2+n+1)^2}{2(2n+1)(n^2-1)}$ M1A1

$= \dfrac{-2n^3 - n^2 + 2n + 1}{2(2n+1)(n^2-1)}$ M1

$= -\dfrac{(n-1)(n+1)(2n+1)}{2(2n+1)(n^2-1)}$ A1

$= -\dfrac{1}{2}$ A1

$\theta = 120°$ AG

[8]

21. (a) PQ = 50 and non-intersecting R1

(b) a construction QT (where T is on the radius MP), parallel to MN,
so that $Q\hat{T}M = 90°$ (angle between tangent and radius = 90°) M1
lengths 50, $x - 10$ and angle θ marked on a diagram, or equivalent R1

Note: Other construction lines are possible.

(c) (i) MN = $\sqrt{50^2 - (x-10)^2}$ A1

(ii) maximum for MN occurs when $x = 10$ A1

(d) (i) $\alpha = 2\pi - 2\theta$ M1

$= 2\pi - 2\arccos\left(\dfrac{x-10}{50}\right)$ A1

(ii) $\beta = 2\pi - \alpha\ (= 2\theta)$ A1

$= 2\left(\cos^{-1}\left(\dfrac{x-10}{50}\right)\right)$ A1

(e) (i) $b(x) = x\alpha + 10\beta + 2\sqrt{50^2 - (x-10)^2}$ A1A1A1

$= x\left(2\pi - 2\left(\cos^{-1}\left(\dfrac{x-10}{50}\right)\right)\right) + 20\left(\cos^{-1}\left(\dfrac{x-10}{50}\right)\right) + 2\sqrt{50^2 - (x-10)^2}$

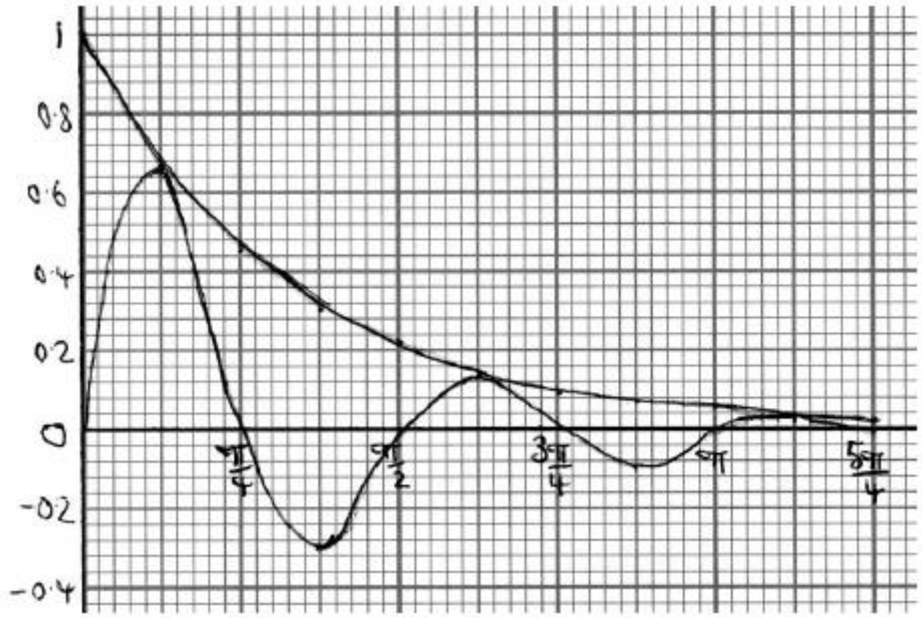
M1A1

(ii) maximum value of perimeter = 276 — A2

(iii) perimeter of 200 cm $b(x) = 200$ — (M1)
when $x = 21.2$ — A1

[18]

22. (a)



A3

Note: Award A1 for each correct **shape**,
A1 for correct relative position.

(b) $e^{-x}\sin(4x) = 0$ — (M1)
$\sin(4x) = 0$ — A1
$4x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$ — A1
$x = 0, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}, \frac{5\pi}{4}$ — AG

(c) $e^{-x} = e^{-x}\sin(4x)$ or reference to graph
$\sin 4x = 1$ — M1
$4x = \frac{\pi}{4}, \frac{5\pi}{2}, \frac{9\pi}{2}$ — A1
$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$ — A1 N3

(d) (i) $y = e^{-x}\sin 4x$

$\frac{dy}{dx} = -e^{-x}\sin 4x + 4e^{-x}\cos 4x$ M1A1

$y = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$ A1

verifying equality of gradients at one point R1

verifying at the other two R1

(ii) since $\frac{dy}{dx} \neq 0$ at these points they cannot be local maxima R1

(e) (i) maximum when $y' = 4e^{-x}\cos 4x - e^{-x}\sin 4x = 0$ M1

$x = \frac{\arctan(4)}{4}, \frac{\arctan(4)+\pi}{4}, \frac{\arctan(4)+2\pi}{4}, \ldots$

maxima occur at

$x = \frac{\arctan(4)}{4}, \frac{\arctan(4)+2\pi}{4}, \frac{\arctan(4)+4\pi}{4}$ A1

so $y_1 = e^{-\frac{1}{4}(\arctan(4))}\sin(\arctan(4))$ (= 0.696) A1

$y_2 = e^{-\frac{1}{4}(\arctan(4)+2\pi)}\sin(\arctan(4)+2\pi)$ A1

$\left(= e^{-\frac{1}{4}(\arctan(4)+2\pi)}\sin(\arctan(4)) = 0.145\right)$

$y_3 = e^{-\frac{1}{4}(\arctan(4)+4\pi)}\sin(\arctan(4)+4\pi)$ A1

$\left(= e^{-\frac{1}{4}(\arctan(4)+4\pi)}\sin(\arctan(4)) = 0.0301\right)$ N3

(ii) for finding and comparing $\frac{y_3}{y_2}$ and $\frac{y_2}{y_1}$ M1

$r = e^{-\frac{\pi}{2}}$ A1

Note: Exact values must be used to gain the M1 and the A1.

[22]

23. (a) shaded area = area of triangle – area of sector, i.e. (M1)

$\left(\frac{1}{2}\times 4^2 \sin x\right) - \left(\frac{1}{2}2^2 x\right) = 8\sin x - 2x$ A1A1AG

(b) **EITHER**

any method from GDC gaining $x \approx 1.32$ (M1)(A1) maximum value for given domain is 5.11 A2

OR

$\frac{dA}{dx} = 8\cos x - 2$ A1

set $\frac{dA}{dx} = 0$, hence $8\cos x - 2 = 0$ M1

$\cos x = \frac{1}{4} \Rightarrow x \approx 1.32$ A1

hence $A_{max} = 5.11$ A1

[7]

24. $\frac{9}{\sin C} = \frac{12}{\sin B}$ (M1)

$\frac{9}{\sin C} = \frac{12}{\sin 2C}$ A1

Using double angle formula $\frac{9}{\sin C} = \frac{12}{2\sin C\cos C}$ M1

$\Rightarrow 9(2\sin C\cos C) = 12\sin C$

$\Rightarrow 6\sin C(3\cos C - 2) = 0$ or equivalent (A1)

$(\sin C \neq 0)$

$\Rightarrow \cos C = \frac{2}{3}$ A1

[5]

25. **METHOD 1**

AC = 5 and AB = $\sqrt{13}$ (may be seen on diagram) (A1)

$\cos\alpha = \frac{3}{5}$ **and** $\sin\alpha = \frac{4}{5}$ (A1)

$\cos\beta = \frac{3}{\sqrt{13}}$ **and** $\sin\beta = \frac{2}{\sqrt{13}}$ (A1)

Note: If only the two cosines are correctly given award (A1)(A1)(A0).

Use of $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ (M1)

$= \frac{3}{5}\times\frac{3}{\sqrt{13}} + \frac{4}{5}\times\frac{2}{\sqrt{13}}$ (substituting) M1

$= \frac{17}{5\sqrt{13}} \left(= \frac{17\sqrt{13}}{65}\right)$ A1 N1

METHOD 2

AC = 5 and AB = $\sqrt{13}$ (may be seen on diagram) (A1)

Use of $\cos(\alpha + \beta) = \frac{AC^2 + AB^2 - BC^2}{2(AC)(AB)}$ (M1)

$= \frac{25 + 13 - 36}{2\times 5\times\sqrt{13}} \left(= \frac{1}{5\sqrt{13}}\right)$ A1

Use of $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$ (M1)

$\cos\alpha = \frac{3}{5}$ **and** $\cos\beta = \frac{3}{\sqrt{13}}$ (A1)

$\cos(\alpha - \beta) = \frac{17}{5\sqrt{13}} \left(= 2\times\frac{3}{5}\times\frac{3}{\sqrt{13}} - \frac{1}{5\sqrt{13}}\right) \left(= \frac{17\sqrt{13}}{65}\right)$ A1 N1

[6]

26. 10 cm water depth corresponds to $16\sec\left(\frac{\pi x}{36}\right) - 32 = -6$ (A1)

Rearranging to obtain an equation of the form $\sec\left(\frac{\pi x}{36}\right) = k$ or equivalent

ie making a trignometrical function the subject of the equation. M1

$\cos\left(\frac{\pi x}{36}\right) = \frac{8}{13}$ (A1)

$\frac{\pi x}{36} = \pm \arccos \frac{8}{13}$ M1

$x = \pm \frac{36}{\pi} \arccos \frac{8}{13}$ A1

Note: Do not penalize the omission of ±.

Width of water surface is $\frac{72}{\pi} \arccos \frac{8}{13}$ (cm) R1 N1

Note: Candidate who starts with 10 instead of –6 has the potential to gain the two M1 marks and the R1 mark.

[6]

27. (a) $y = \arccos(1.2 - \cos x)$ A1

$y = \arcsin(1.4 - \sin x)$ A1

(b) The solutions are

$x = 1.26, y = 0.464$ A1A1

$x = 0.464, y = 1.26$ A1A1

[6]

28. $(\sin\theta + \mathrm{i}\,(1 - \cos\theta))^2 = \sin^2\theta - (1 - \cos\theta)^2 + \mathrm{i}\,2\sin\theta(1 - \cos\theta)$ M1A1

Let α be the required argument.

$$\tan\alpha = \frac{2\sin\theta(1-\cos\theta)}{\sin^2\theta-(1-\cos\theta)^2}$$ M1

$$= \frac{2\sin\theta(1-\cos\theta)}{(1-\cos^2\theta)-(1-2\cos\theta+\cos^2\theta)}$$ (M1)

$$= \frac{2\sin\theta(1-\cos\theta)}{2\cos\theta(1-\cos\theta)}$$ A1

$= \tan\theta$ A1

$\alpha = \theta$ A1

[7]

29. (a) CD = AC – AD = $b - c\cos A$ R1AG

(b) **METHOD 1**

$BC^2 = BD^2 + CD^2$ (M1)

$a^2 = (c\sin A)^2 + (b - c\cos A)^2$ (A1)

$= c^2\sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A$ A1

$= b^2 + c^2 - 2bc\cos A$ A1

METHOD 2

$BD^2 = AB^2 - AD^2 = BC^2 - CD^2$ (M1)(A1)

$\Rightarrow c^2 - c^2\cos^2 A = a^2 - b^2 + 2bc\cos A - c^2\cos^2 A$ A1

$\Rightarrow a^2 = b^2 + c^2 - 2bc\cos A$ A1

(c) **METHOD 1**

$b^2 = a^2 + c^2 - 2ac\cos 60° \Rightarrow b^2 = a^2 + c^2 - ac$ (M1)A1

$\Rightarrow c^2 - ac + a^2 - b^2 = 0$ M1

$\Rightarrow c = \dfrac{a \pm \sqrt{(-a)^2 - 4(a^2 - b^2)}}{2}$ (M1)A1

$= \dfrac{a \pm \sqrt{4b^2 - 3a^2}}{2} = \dfrac{a}{2} \pm \sqrt{\dfrac{4b^2 - 3a^2}{4}}$ (M1)A1

$= \dfrac{1}{2}a \pm \sqrt{b^2 - \dfrac{3}{4}a^2}$ AG

Note: Candidates can only obtain a maximum of the first three marks if they **verify** that the answer given in the question satisfies the equation.

METHOD 2

$b^2 = a^2 + c^2 - 2ac\cos 60° \Rightarrow b^2 = a^2 + c^2 - ac$ (M1)A1

$c^2 - ac = b^2 - a^2$ (M1)

$c^2 - ac + \left(\dfrac{a}{2}\right)^2 = b^2 - a^2 + \left(\dfrac{a}{2}\right)^2$ M1A1

$\left(c - \dfrac{a}{2}\right)^2 = b^2 - \dfrac{3}{4}a^2$ (A1)

$c - \dfrac{a}{2} = \pm\sqrt{b^2 - \dfrac{3}{4}a^2}$ A1

$\Rightarrow c = \dfrac{1}{2}a \pm \sqrt{b^2 - \dfrac{3}{4}a^2}$ AG

[12]

30. PR = $h\tan 55°$, QR = $h\tan 50°$ where RS = h M1A1A1

Use the cosine rule in triangle PQR. (M1)

$20^2 = h^2\tan^2 55° + h^2\tan^2 50° - 2h\tan 55°\, h\tan 50°\cos 45°$ A1

$h^2 = \dfrac{400}{\tan^2 55° + \tan^2 50° - 2\tan 55°\tan 50°\cos 45°}$ (A1)

= 379.9... (A1)

h = 19.5 (m) A1

[8]

31. (a) Either finding depths graphically, using $\sin\dfrac{\pi t}{6} = \pm 1$

or solving $h'(t) = 0$ for t (M1)

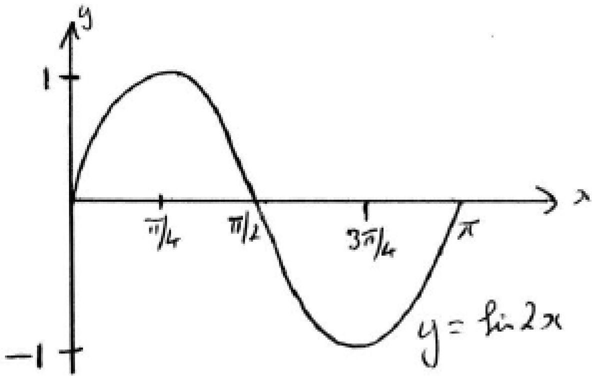
$h(t)_{\max} = 12$ (m), $h(t)_{\min} = 4$ (m) A1A1 N3

(b) Attempting to solve $8 + 4\sin\frac{\pi t}{6} = 8$ algebraically or graphically (M1)

$t \in [0, 6] \cup [12, 18] \cup \{24\}$ A1A1 N3

[6]

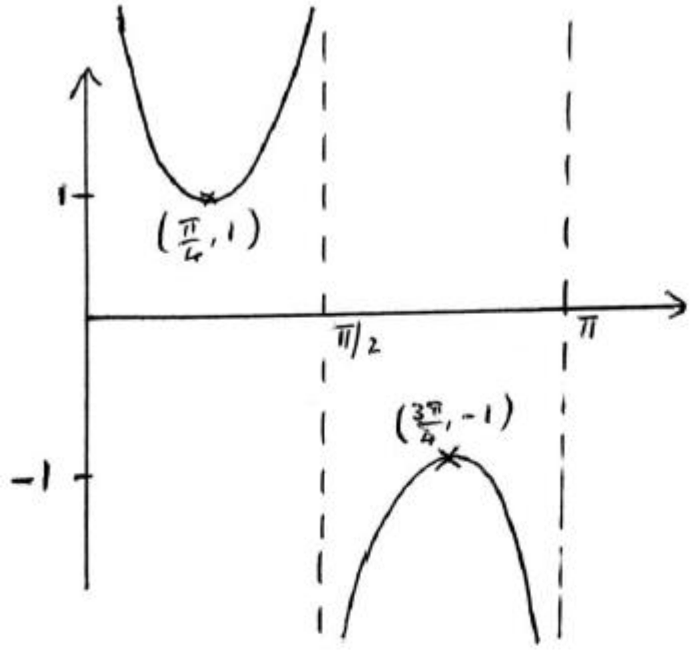
32. (a)



A2

Note: Award A1 for shape.
A1 for scales given on each axis.

(b)



A5

Asymptotes $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$

Max$\left(\frac{3\pi}{4}, -1\right)$, Min$\left(\frac{\pi}{4}, 1\right)$

Note: Award A1 for shape
A2 for asymptotes, A1 for one error, A0 otherwise.
A1 for max.
A1 for min.

(c) $\tan x + \cot x \equiv \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$ M1

$\equiv \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$ A1

$\equiv \frac{1}{\frac{1}{2}\sin 2x}$ A1

$\equiv 2 \csc 2x$ AG

(d) $\tan 2x + \cot 2x \equiv 2 \csc 4x$ (M1)

Max is at $\left(\frac{3\pi}{2}, -2\right)$ A1A1

Min is at $\left(\frac{\pi}{8}, 2\right)$ A1A1

(e) $\csc 2x = 1.5 \tan x - 0.5$

$\frac{1}{2}\tan x + \frac{1}{2}\cot x = \frac{3}{2}\tan x - \frac{1}{2}$ M1

$\tan x + \cot x = 3 \tan x - 1$

$2 \tan x - \frac{1}{\tan x} - 1 = 0$ M1

$2 \tan^2 x - \tan x - 1 = 0$ A1

$(2 \tan x + 1)(\tan x - 1) = 0$ M1

$\tan x = -\frac{1}{2}$ or 1 A1

$x = \frac{\pi}{4}$ A1

Note: Award A0 for answer in degrees or if more than one value given for *x*.

[21]

33. $\frac{\sin B}{6.5} = \frac{\sin 35°}{4}$ M1

$\hat{B} = 68.8°$ or $111°$ A1A1

$\hat{C} = 76.2°$ or $33.8°$ (accept 34°) A1

$\frac{\text{AB}}{\sin \text{C}} = \frac{\text{BC}}{\sin A}$

$\frac{\text{AB}}{\sin 76.2°} = \frac{4}{\sin 35°}$ (M1)

AB = 6.77 cm

$\frac{\text{AB}}{\sin 33.8°} = \frac{4}{\sin 35°}$ A1

AB = 3.88cm (accept 3.90) A1

[7]

34. $2 \sin x \cos x - \sqrt{2} \cos x = 0$ (M1)

$\cos x (2 \sin x - \sqrt{2}) = 0$ (A1)

$\cos x = 0$ $\quad$ $\sin x = \frac{\sqrt{2}}{2}$ A1

$x = \frac{\pi}{2}$ $\quad$ $x = \frac{\pi}{4}, \frac{3\pi}{4}$ A1A1A1

[6]

35. (a) $\sin B = \frac{5}{13}$ A1

(b) $\cos B = -\frac{12}{13}$ A1

(c) $\sin 2B = 2\sin B \cos B$ (M1)

$= 2 \times \frac{5}{13} \times -\frac{12}{13}$

$= -\frac{120}{169}$ A1

(d) $\cos 2B = 2\cos^2 B - 1$ (M1)

$= 2\left(\frac{144}{169}\right) - 1$

$= \frac{119}{169}$ A1

[6]

36. Using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ (M1)

$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{3}{4}$

$3\tan^2\theta + 8\tan\theta - 3 = 0$ A1

Using factorisation or the quadratic formula (M1)

$\tan\theta = \frac{1}{3}$ or -3 A1A1

[5]

37. (a) $\sin x \cos\alpha - \cos x \sin\alpha = k \sin x \cos\alpha + k \cos x \sin\alpha$ (M1)

$\Rightarrow \tan x \cos\alpha - \sin\alpha = k \tan x \cos\alpha + k \sin\alpha$ M1

$\Rightarrow \tan x = \frac{-(k+1)\sin\alpha}{(k-1)\cos\alpha} \left(= \frac{-(k+1)}{(k-1)} \tan\alpha \right)$ A1

(b) $\tan x = \dfrac{-\frac{3}{2}\sin 210°}{-\frac{1}{2}\cos 210°}$ (M1)

Now $\sin 210° = -\sin 30° = -\frac{1}{2}$ and $\cos 210° = -\cos 30° = -\frac{\sqrt{3}}{2}$ A1A1

$\tan x = \dfrac{3\times -\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{3}{2}\times\frac{2}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$ A1

$\Rightarrow\ x = 60°, 240°$ A1A1

[9]

38. $2\tan^2\theta - 5\sec\theta - 10 = 0$

Using $1 + \tan^2\theta = \sec^2\theta$, $\Rightarrow$ $2(\sec^2\theta - 1) - 5\sec\theta - 10 = 0$ (M1)

$2\sec^2\theta - 5\sec\theta - 12 = 0$ A1

Solving the equation e.g. $(2\sec\theta + 3)(\sec\theta - 4) = 0$ (M1)

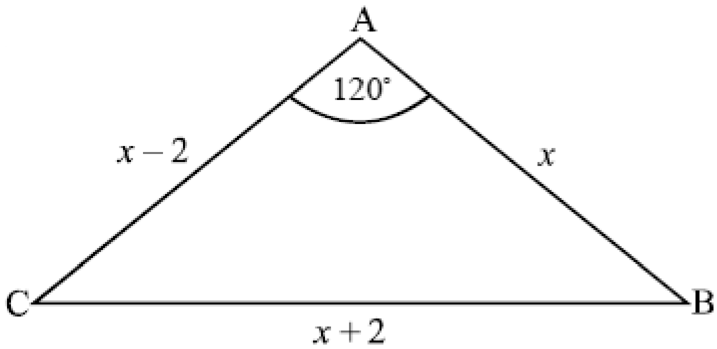
$\sec\theta = -\frac{3}{2}$ or $\sec\theta = 4$ A1

θ in second quadrant $\Rightarrow$ $\sec\theta$ is negative (R1)

$\Rightarrow$ $\sec\theta = -\frac{3}{2}$ A1 N3

[6]

39. (a)



(M1)

$(x + 2)^2 = (x - 2)^2 + x^2 - 2(x - 2)\,x\cos 120°$ M1A1

$x^2 + 4x + 4 = x^2 - 4x + 4 + x^2 + x^2 - 2x$ (M1)

$0 = 2x^2 - 10x$ A1

$0 = x(x - 5)$

$x = 5$ A1

(b) Area = $\frac{1}{2} \times 5 \times 3 \times \sin 120°$ M1A1

$= \frac{1}{2} \times 15 \times \frac{\sqrt{3}}{2}$ A1

$= \frac{15\sqrt{3}}{4}$ AG

(c) $\sin A = \frac{\sqrt{3}}{2}$

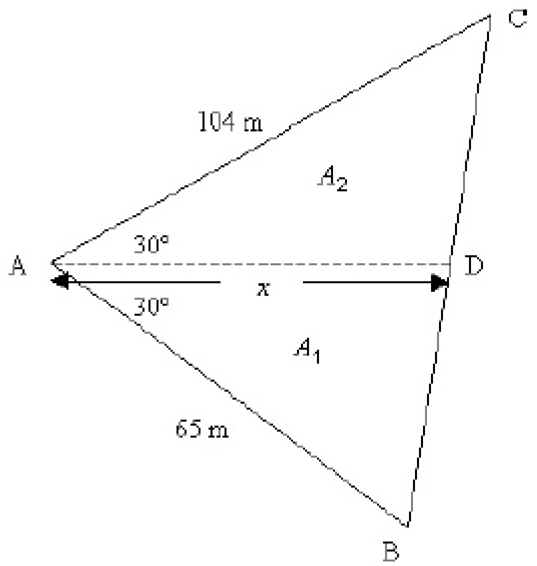
$\frac{15\sqrt{3}}{4} = \frac{1}{2} \times 5 \times 7 \times \sin B \Rightarrow \sin B = \frac{3\sqrt{3}}{14}$ M1A1

Similarly $\sin C = \frac{5\sqrt{3}}{14}$ A1

$\sin A + \sin B + \sin C = \frac{15\sqrt{3}}{14}$ A1

[13]

40.



(a) Using the cosine rule ($a^2 = b^2 + c^2 - 2bc \cos A$) (M1)
Substituting correctly
$BC^2 = 65^2 + 104^2 - 2\,(65)\,(104) \cos 60°$ A1
$= 4225 + 10\,816 - 6760 = 8281$
$\Rightarrow$ BC = 91m A1 N2

(b) Finding the area using = $\frac{1}{2}bc\sin A$ (M1)

Substituting correctly, area = $\frac{1}{2}(65)(104)\sin 60°$ A1

= $1690\sqrt{3}$ (accept p = 1690) A1 N2

(c) (i) Smaller area $A_1 = \left(\frac{1}{2}\right)(65)(x)\sin 30°$ (M1)A1

$= \frac{65x}{4}$ AG N0

Larger area $A_2 = \left(\frac{1}{2}\right)(104)(x)\sin 30°$ M1

$= 26x$ A1 N1

(ii) Using $A_1 + A_2 = A$ (M1)

Substituting $\frac{65x}{4} + 26x = 1690\sqrt{3}$ A1

Simplifying $\frac{169x}{4} = 1690\sqrt{3}$ A1

Solving $x = \frac{4 \times 1690\sqrt{3}}{169}$

$\Rightarrow x = 40\sqrt{3}$ (accept q = 40) A1 N1

(d) Using sin rule in ΔADB and ΔACD (M1)

Substituting correctly $\frac{\text{BD}}{\sin 30°} = \frac{65}{\sin \text{A}\hat{\text{D}}\text{B}} \Rightarrow \frac{\text{BD}}{65} = \frac{\sin 30°}{\sin \text{A}\hat{\text{D}}\text{B}}$ A1

and $\frac{\text{DC}}{\sin 30°} = \frac{104}{\sin \text{A}\hat{\text{D}}\text{C}} \Rightarrow \frac{\text{DC}}{104} = \frac{\sin 30°}{\sin \text{A}\hat{\text{D}}\text{C}}$ A1

Since $\text{A}\hat{\text{D}}\text{B} + \text{A}\hat{\text{D}}\text{C} = 180°$ R1

It follows that $\sin \text{A}\hat{\text{D}}\text{B} = \sin \text{A}\hat{\text{D}}\text{C}$ R1

$\frac{\text{BD}}{65} = \frac{\text{DC}}{104} \Rightarrow \frac{\text{BD}}{\text{DC}} = \frac{65}{104}$ A1

$\Rightarrow \frac{\text{BD}}{\text{DC}} = \frac{5}{8}$ AG N0

[20]

41. (a) $4(1-2s^2)-3s\frac{1}{s^3}+6=0$ M1A1

$4s^2-8s^4+6s^2-3=0$ A1

$8s^4-10s^2+3=0$ AG

(b) Attempt to factorise or use the quadratic formula (M1)

$\sin^2 x = \frac{1}{2}$ or $\sin^2 x = \frac{3}{4}$ (A1)

$\sin x = \frac{\sqrt{2}}{2} \Rightarrow x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$ A1A1

$\sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3}$ or $x = \frac{2\pi}{3}$ A1A1

Note: Penalise A1 if extraneous solutions given.

[9]

42. **METHOD 1**

Attempting to use the cosine rule i.e.

$BC^2 = AB^2 + AC^2 - 2 \times AB \times AC \times \cos\ B\hat{A}C$ (M1)

$6^2 = 8.75^2 + AC^2 - 2 \times 8.75 \times AC \times \cos 37.8°$ (or equivalent) A1

Attempting to solve the quadratic in AC e.g. graphically, numerically or with quadratic formula M1A1

Evidence from a sketch graph or their quadratic formula (AC =…) that there are two values of AC to determine. (A1)

AC = 9.60 or AC = 4.22 A1A1 N4

Note: Award (M1)A1M1A1(A0)A1A0 for one correct value of AC.

METHOD 2

Attempting to use the sine rule i.e. $\frac{\text{BC}}{\sin \text{B}\hat{\text{A}}\text{C}} = \frac{\text{AB}}{\sin \text{A}\hat{\text{C}}\text{B}}$ (M1)

$\sin C = \frac{8.75 \sin 37.8°}{6}$ $(= 0.8938...)$ (A1)

C = 63.3576... ° A1

C = 116.6423... ° **and** B = 78.842... ° or B = 25.5576... ° A1

EITHER

Attempting to solve $\frac{\text{AC}}{\sin 78.842...°} = \frac{6}{\sin 37.8°}$ or

$\frac{\text{AC}}{\sin 25.5576...°} = \frac{6}{\sin 37.8°}$ M1

OR

Attempting to solve $\text{AC}^2 = 8.75^2 + 6^2 - 2 \times 8.75 \times 6 \times \cos 25.5576...°$ or

$\text{AC}^2 = 8.75^2 + 6^2 - 2 \times 8.75 \times 6 \times \cos 78.842...°$ M1

AC = 9.60 or AC = 4.22 A1A1 N4

Note: Award (M1)(A1)A1A0M1A1A0 for one correct value of AC.

[7]

43. $\sin\left(x + \frac{\pi}{3}\right) = \sin x \cos\left(\frac{\pi}{3}\right) + \cos x \sin\left(\frac{\pi}{3}\right)$ (M1)

$\sin x \cos\left(\frac{\pi}{3}\right) + \cos x \sin\left(\frac{\pi}{3}\right) = 2 \sin x \sin\left(\frac{\pi}{3}\right)$

$\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x = 2 \times \frac{\sqrt{3}}{2} \sin x$ A1

dividing by cos x and rearranging M1

$\tan x = \frac{\sqrt{3}}{2\sqrt{3} - 1}$ A1

rationalizing the denominator M1

$11 \tan x = 6 + \sqrt{3}$ A1

[6]

44. **METHOD 1**

$\frac{\sin C}{7} = \frac{\sin 40}{5}$ M1(A1)

$B\hat{C}D = 64.14...°$ A1

CD = 2 × 5cos 64.14... M1

Note: A1so allow use of sine or cosine rule.

CD = 4.36 A1

METHOD 2

let AC = x

cosine rule

$5^2 = 7^2 + x^2 - 2 \times 7 \times x \cos 40$ M1A1

$x^2 - 10.7... x + 24 = 0$

$x = \frac{10.7... \pm \sqrt{(10.7...)^2 - 4 \times 24}}{2}$ (M1)

$x = 7.54; 3.18$ (A1)

CD is the difference in these two values = 4.36 A1

Note: Other methods may be seen.

[5]