MATH HL2 EXAM PREP – CORE TOPICS – CIRCULAR FUNCTIONS & TRIG

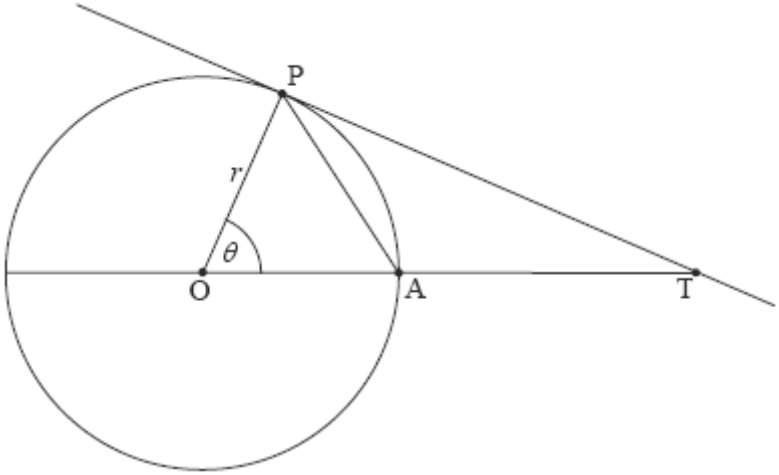
1. (a) Show that $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan \theta$.

(2)

(b) Hence find the value of $\cot\frac{\pi}{8}$ in the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$.

(3)
(Total 5 marks)

2. The diagram shows a tangent, (TP), to the circle with centre O and radius r. The size of $P\hat{O}A$ is θ radians.



(a) Find the area of triangle AOP in terms of r and θ.

(1)

(b) Find the area of triangle POT in terms of r and θ.

(2)

(c) Using your results from part (a) and part (b), show that $\sin\theta < \theta < \tan\theta$.

(2)
(Total 5 marks)

3. The radius of the circle with centre C is 7 cm and the radius of the circle with centre D is 5 cm. If the length of the chord [AB] is 9 cm, find the area of the shaded region enclosed by the two arcs AB.

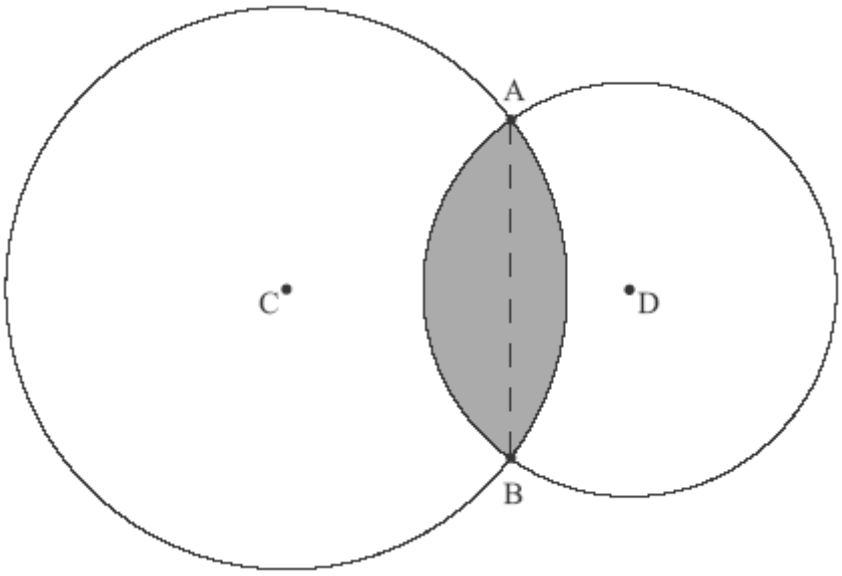


diagram not to scale

(Total 7 marks)

4. The points P and Q lie on a circle, with centre O and radius 8 cm, such that $P\hat{O}Q = 59°$.

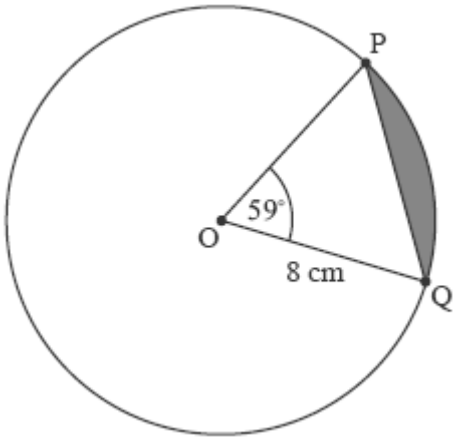


diagram not to scale

Find the area of the shaded segment of the circle contained between the arc PQ and the chord [PQ].

(Total 5 marks)

5. The vertices of an equilateral triangle, with perimeter P and area A, lie on a circle with radius r.

Find an expression for $\frac{P}{A}$ in the form $\frac{k}{r}$, where $k \in \mathbb{Z}^+$.

(Total 6 marks)

6. (a) Given that $\alpha > 1$, use the substitution $u = \frac{1}{x}$ to show that

$$\int_1^{\alpha} \frac{1}{1+x^2}\mathrm{d}x = \int_{\frac{1}{\alpha}}^{1} \frac{1}{1+u^2}\mathrm{d}u .$$

(5)

(b) **Hence** show that $\arctan \alpha + \arctan\frac{1}{\alpha} = \frac{\pi}{2}$.

(2)
(Total 7 marks)

7. (a) Show that $\sin 2nx = \sin((2n + 1)x)\cos x - \cos((2n + 1)x)\sin x$.

(2)

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + ... + \cos((2n - 1)x) = \frac{\sin 2nx}{2\sin x},$$

for all $n \in \mathbb{Z}^+$, $\sin x \neq 0$.

(12)

(c) Solve the equation $\cos x + \cos 3x = \frac{1}{2}$, $0 < x < \pi$.

(6)
(Total 20 marks)

8. Throughout this question x satisfies $0 \leq x < \frac{\pi}{2}$.

(a) Solve the differential equation $\sec^2 x \ \frac{dy}{dx} = -y^2$, where $y = 1$ when $x = 0$.

Give your answer in the form $y = f(x)$.

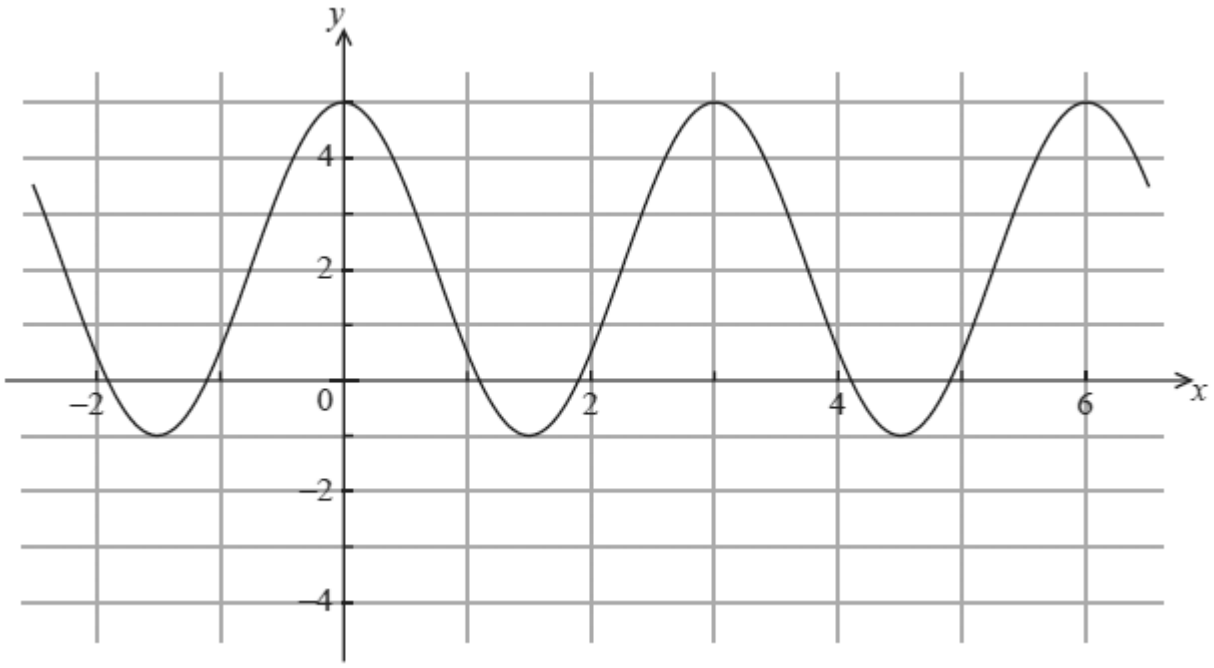
(7)

(b) (i) Prove that $1 \leq \sec x \leq 1 + \tan x$.

(ii) Deduce that $\frac{\pi}{4} \leq \int_0^{\frac{\pi}{4}} \sec x dx \leq \frac{\pi}{4} + \frac{1}{2} \ln 2$.

(8)
(Total 15 marks)

9. The graph below shows $y = a \cos (bx) + c$.



Find the value of a, the value of b and the value of c.

(Total 4 marks)

10. In the right circular cone below, O is the centre of the base which has radius 6 cm.
The points B and C are on the circumference of the base of the cone. The height AO of the cone is 8 cm and the angle $B\hat{O}C$ is 60°.

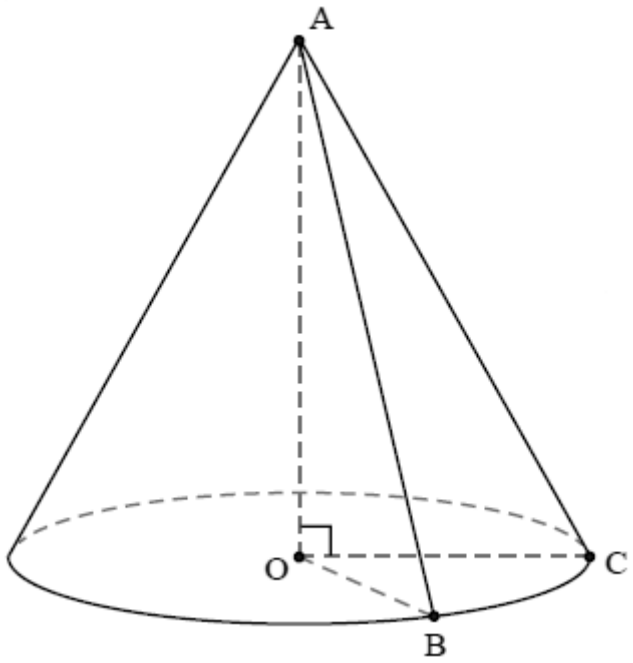


diagram not to scale

Calculate the size of the angle $B\hat{A}C$.

(Total 6 marks)

11. Consider the triangle ABC where $B\hat{A}C$ = 70°, AB = 8 cm and AC = 7 cm. The point D on the side BC is such that $\frac{BD}{DC} = 2$.
Determine the length of AD.

(Total 6 marks)

12. The interior of a circle of radius 2 cm is divided into an infinite number of sectors.
The areas of these sectors form a geometric sequence with common ratio *k*. The angle of the first sector is θ radians.

(a) Show that $\theta = 2\pi(1 - k)$.

(5)

(b) The perimeter of the third sector is half the perimeter of the first sector.

Find the value of k and of θ.

(7)
(Total 12 marks)

13. Consider the function $f : x \to \sqrt{\frac{\pi}{4} - \arccos x}$.

(a) Find the largest possible domain of f.

(4)

(b) Determine an expression for the inverse function, f^{-1}, and write down its domain.

(4)
(Total 8 marks)

14. Let α be the angle between the unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, where $0 \leq \alpha \leq \pi$.

(a) Express $|\boldsymbol{a} - \boldsymbol{b}|$ and $|\boldsymbol{a} + \boldsymbol{b}|$ in terms of α.

(3)

(b) **Hence** determine the value of $\cos \alpha$ for which $|\boldsymbol{a} + \boldsymbol{b}| = 3|\boldsymbol{a} - \boldsymbol{b}|$.

(2)
(Total 5 marks)

15. (a) A particle P moves in a straight line with displacement relative to origin given by

$$s = 2 \sin(\pi t) + \sin(2\pi t),\ t \geq 0,$$

where t is the time in seconds and the displacement is measured in centimetres.

(i) Write down the period of the function s.

(ii) Find expressions for the velocity, v, and the acceleration, a, of P.

(iii) Determine all the solutions of the equation $v = 0$ for $0 \leq t \leq 4$.

(10)

(b) Consider the function

$$f(x) = A \sin(ax) + B \sin(bx), A, a, B, b, x \in \mathbb{R}.$$

Use mathematical induction to prove that the $(2n)^{\text{th}}$ derivative of f is given by $f^{(2n)}(x) = (-1)^n (Aa^{2n} \sin(ax) + Bb^{2n} \sin(bx))$, for all $n \in \mathbb{Z}^+$.

(8)
(Total 18 marks)

16. Triangle ABC has AB = 5cm, BC = 6 cm and area 10 cm^2.

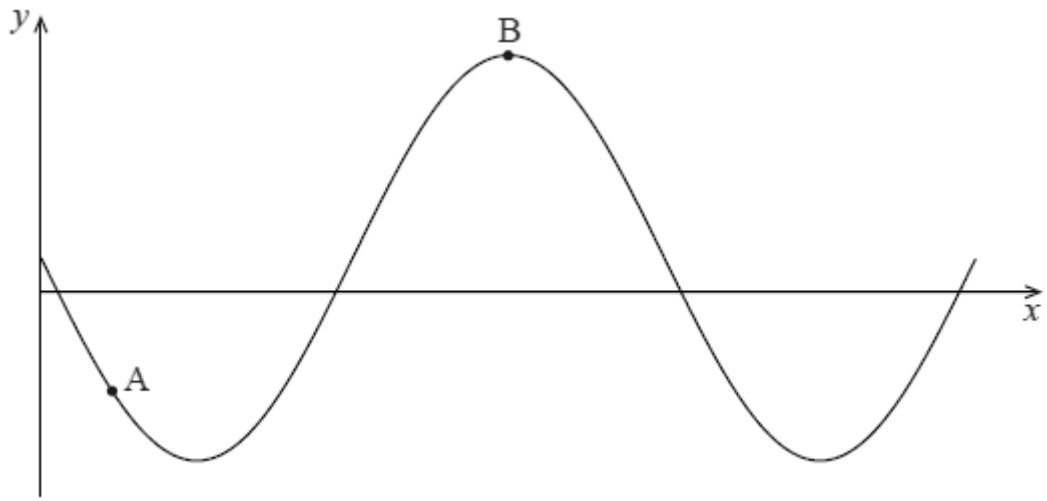
(a) Find $\sin \hat{B}$.

(2)

(b) **Hence**, find the two possible values of AC, giving your answers correct to two decimal places.

(4)
(Total 6 marks)

17. The diagram below shows a curve with equation $y = 1 + k \sin x$, defined for $0 \leq x \leq 3\pi$.



The point $A\left(\frac{\pi}{6}, -2\right)$ lies on the curve and B(a, b) is the maximum point.

(a) Show that $k = -6$.

(2)

(b) Hence, find the values of a and b.

(3)
(Total 5 marks)

18. (a) Show that $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$.

(2)

(b) Hence, or otherwise, find the value of arctan (2) + arctan (3).

(3)
(Total 5 marks)

19. The diagram below shows two straight lines intersecting at O and two circles, each with centre O. The outer circle has radius R and the inner circle has radius r.

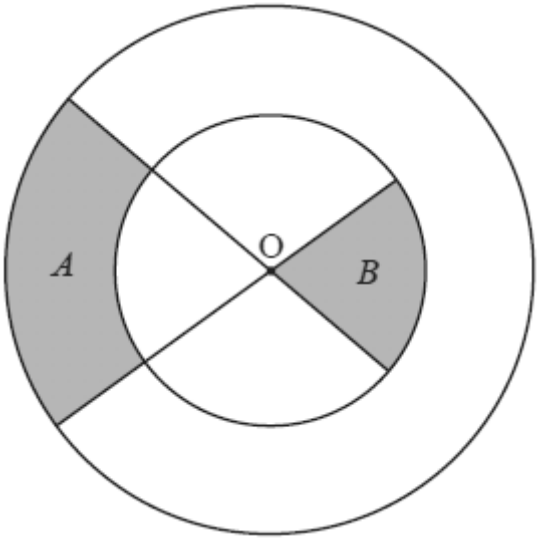


diagram not to scale

Consider the shaded regions with areas A and B. Given that $A : B = 2 : 1$, find the **exact** value of the ratio $R : r$.

(Total 5 marks)

20. A triangle has sides of length $(n^2 + n + 1)$, $(2n + 1)$ and $(n^2 - 1)$ where $n > 1$.

(a) Explain why the side $(n^2 + n + 1)$ must be the longest side of the triangle.

(3)

(b) Show that the largest angle, θ, of the triangle is 120°.

(5)

(Total 8 marks)

21. Two non-intersecting circles C_1, containing points M and S, and C_2, containing points N and R, have centres P and Q where PQ = 50. The line segments [MN] and [SR] are common tangents to the circles. The size of the reflex angle MPS is α, the size of the obtuse angle NQR is β, and the size of the angle MPQ is θ. The arc length MS is l_1 and the arc length NR is l_2. This information is represented in the diagram below.

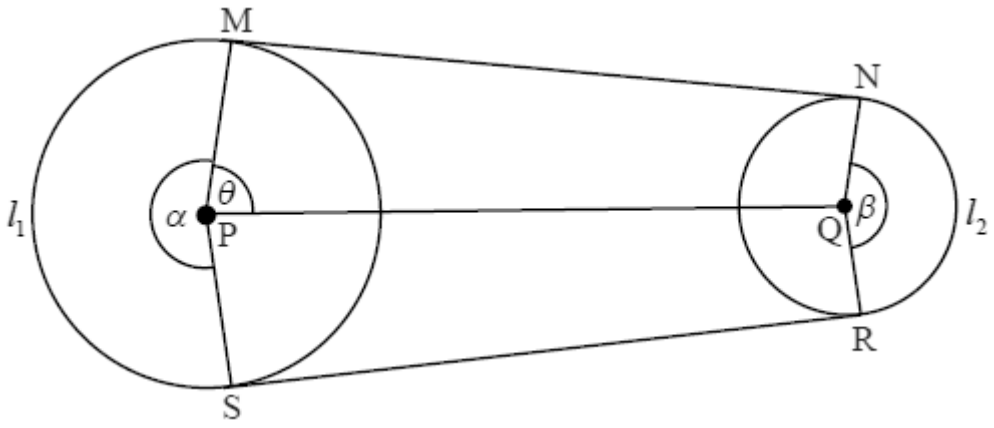


diagram not to scale

The radius of C_1 is x, where $x \geq 10$ and the radius of C_2 is 10.

(a) Explain why $x < 40$.

(1)

(b) Show that $\cos\theta = \dfrac{x-10}{50}$.

(2)

(c) (i) Find an expression for MN in terms of x.

(ii) Find the value of x that maximises MN.

(2)

(d) Find an expression in terms of x for

(i) α;

(ii) β.

(4)

(e) The length of the perimeter is given by $l_1 + l_2 + \text{MN} + \text{SR}$.

(i) Find an expression, $b(x)$, for the length of the perimeter in terms of x.

(ii) Find the maximum value of the length of the perimeter.

(iii) Find the value of x that gives a perimeter of length 200.

(9)

(Total 18 marks)

22. Consider the graphs $y = e^{-x}$ and $y = e^{-x} \sin 4x$, for $0 \leq x \leq \frac{5\pi}{4}$.

(a) On the same set of axes draw, on graph paper, the graphs, for $0 \leq x \leq \frac{5\pi}{4}$.

Use a scale of 1 cm to $\frac{\pi}{8}$ on your x-axis and 5 cm to 1 unit on your y-axis.

(3)

(b) Show that the x-intercepts of the graph $y = e^{-x} \sin 4x$ are $\frac{n\pi}{4}$, $n = 0, 1, 2, 3, 4, 5$.

(3)

(c) Find the x-coordinates of the points at which the graph of $y = e^{-x} \sin 4x$ meets the graph of $y = e^{-x}$. Give your answers in terms of π.

(3)

(d) (i) Show that when the graph of $y = e^{-x} \sin 4x$ meets the graph of $y = e^{-x}$, their gradients are equal.

(ii) Hence explain why these three meeting points are not local maxima of the graph $y = e^{-x} \sin 4x$.

(6)

(e) (i) Determine the y-coordinates, y_1, y_2 and y_3, where $y_1 > y_2 > y_3$, of the local maxima of $y = e^{-x} \sin 4x$ for $0 \leq x \leq \frac{5\pi}{4}$. You do not need to show that they are maximum values, but the values should be simplified.

(ii) Show that y_1, y_2 and y_3 form a geometric sequence and determine the common ratio r.

(7)
(Total 22 marks)

23. The diagram below shows two concentric circles with centre O and radii 2 cm and 4 cm. The points P and Q lie on the larger circle and $\hat{POQ} = x$, where $0 < x < \frac{\pi}{2}$.

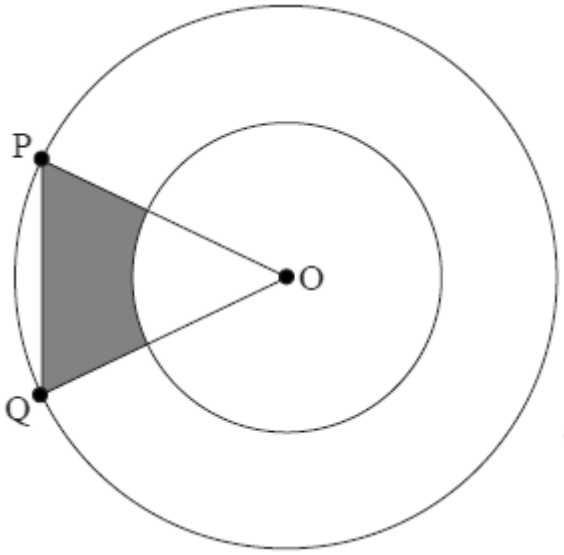


diagram not to scale

(a) Show that the area of the shaded region is $8 \sin x - 2x$.

(3)

(b) Find the maximum area of the shaded region.

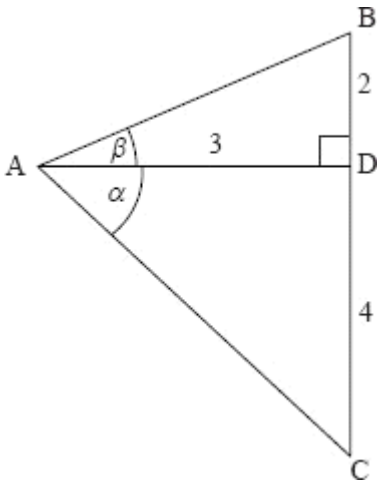
(4)
(Total 7 marks)

24. In triangle ABC, AB = 9 cm, AC =12 cm, and $\hat{B}$ is twice the size of $\hat{C}$.

Find the cosine of $\hat{C}$.

(Total 5 marks)

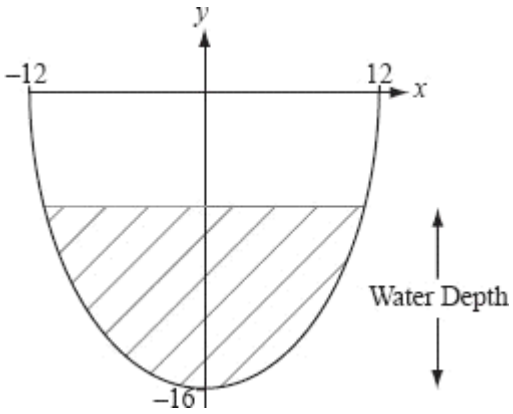
25. In the diagram below, AD is perpendicular to BC.
CD = 4, BD = 2 and AD = 3. C$\hat{A}$D $= \alpha$ and B$\hat{A}$D $= \beta$.



Find the exact value of $\cos(\alpha - \beta)$.

(Total 6 marks)

26. The diagram below shows the boundary of the cross-section of a water channel.



The equation that represents this boundary is $y = 16 \sec\left(\frac{\pi x}{36}\right) - 32$ where x and y are both measured in cm.

The top of the channel is level with the ground and has a width of 24 cm. The maximum depth of the channel is 16 cm.

Find the width of the water surface in the channel when the water depth is 10 cm. Give your answer in the form $a \arccos b$ where $a, b \in \mathbb{R}$.

(Total 6 marks)

27. A system of equations is given by

$$\cos x + \cos y = 1.2$$
$$\sin x + \sin y = 1.4.$$

(a) For each equation express y in terms of x.

(2)

(b) **Hence** solve the system for $0 < x < \pi$, $0 < y < \pi$.

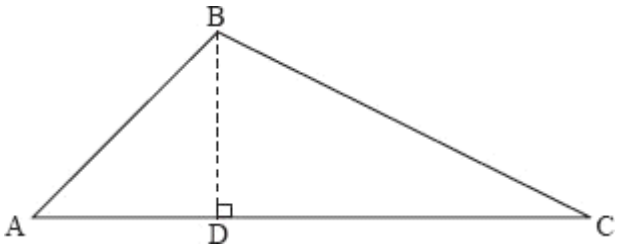
(4)

(Total 6 marks)

28. Find, in its simplest form, the argument of $(\sin\theta + \mathrm{i}\,(1-\cos\theta))^2$ where θ is an acute angle.

(Total 7 marks)

29. In triangle ABC, BC = a, AC = b, AB = c and [BD] is perpendicular to [AC].



(a) Show that CD = $b - c\cos A$.

(1)

(b) **Hence**, by using Pythagoras' Theorem in the triangle BCD, prove the cosine rule for the triangle ABC.

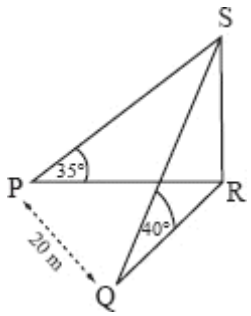
(4)

(c) If $A\hat{B}C = 60°$, use the cosine rule to show that $c = \frac{1}{2}a \pm \sqrt{b^2 - \frac{3}{4}a^2}$.

(7)

(Total 12 marks)

30.



The above three dimensional diagram shows the points P and Q which are respectively west and south-west of the base R of a vertical flagpole RS on horizontal ground. The angles of elevation of the top S of the flagpole from P and Q are respectively 35° and 40°, and PQ = 20 m.

Determine the height of the flagpole.

(Total 8 marks)

31. The depth, $h(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by

$$h(t) = 8 + 4\sin\left(\frac{\pi t}{6}\right), 0 \le t \le 24.$$

(a) Find the maximum depth and the minimum depth of the water.

(3)

(b) Find the values of t for which $h(t) \ge 8$.

(3)
(Total 6 marks)

32. (a) Sketch the curve $f(x) = \sin 2x$, $0 \le x \le \pi$.

(2)

(b) Hence sketch on a separate diagram the graph of $g(x) = \csc 2x$, $0 \le x \le \pi$, clearly stating the coordinates of any local maximum or minimum points and the equations of any asymptotes.

(5)

(c) Show that $\tan x + \cot x \equiv 2 \csc 2x$.

(3)

(d) Hence or otherwise, find the coordinates of the local maximum and local minimum points on the graph of $y = \tan 2x + \cot 2x$, $0 \le x \le \frac{\pi}{2}$.

(5)

(e) Find the solution of the equation $\csc 2x = 1.5 \tan x - 0.5$, $0 \le x \le \frac{\pi}{2}$.

(6)
(Total 21 marks)

33. In a triangle ABC, $\hat{A} = 35°$, BC = 4 cm and AC = 6.5 cm. Find the possible values of $\hat{B}$ and the corresponding values of AB.

(Total 7 marks)

34. Solve $\sin 2x = \sqrt{2} \cos x$, $0 \le x \le \pi$.

(Total 6 marks)

35. The obtuse angle B is such that $\tan B = -\frac{5}{12}$. Find the values of

(a) $\sin B$;

(1)

(b) $\cos B$;

(1)

(c) $\sin 2B$;

(2)

(d0 $\cos 2B$.

(2)

(Total 6 marks)

36. Given that $\tan 2\theta = \frac{3}{4}$, find the possible values of $\tan \theta$.

(Total 5 marks)

37. (a) If $\sin (x - \alpha) = k \sin (x + \alpha)$ express $\tan x$ in terms of k and α.

(3)

(b) Hence find the values of x between 0° and 360° when $k = \frac{1}{2}$ and $\alpha = 210°$.

(6)

(Total 9 marks)

38. The angle θ satisfies the equation $2 \tan^2 \theta - 5 \sec \theta - 10 = 0$, where θ is in the second quadrant. Find the value of $\sec \theta$.

(Total 6 marks)

39. The lengths of the sides of a triangle ABC are $x - 2$, x and $x + 2$. The largest angle is 120°.

(a) Find the value of x.

(6)

(b) Show that the area of the triangle is $\frac{15\sqrt{3}}{4}$.

(3)

(c) Find $\sin A + \sin B + \sin C$ giving your answer in the form $\frac{p\sqrt{q}}{r}$ where $p, q, r \in \mathbb{Z}$.

(4)

(Total 13 marks)

40. A farmer owns a triangular field ABC. The side [AC] is 104 m, the side [AB] is 65 m and the angle between these two sides is 60°.

(a) Calculate the length of the third side of the field.

(3)

(b) Find the area of the field in the form $p\sqrt{3}$, where p is an integer.

(3)

Let D be a point on [BC] such that [AD] bisects the 60° angle. The farmer divides the field into two parts by constructing a straight fence [AD] of length x metres.

(c) (i) Show that the area o the smaller part is given by $\frac{65x}{4}$ and find an expression for the area of the larger part.

(ii) Hence, find the value of x in the form $q\sqrt{3}$, where q is an integer.

(8)

(d) Prove that $\frac{\text{BD}}{\text{DC}} = \frac{5}{8}$.

(6)
(Total 20 marks)

41. Let $\sin x = s$.

(a) Show that the equation $4 \cos 2x + 3 \sin x \operatorname{cosec}^3 x + 6 = 0$ can be expressed as $8s^4 - 10s^2 + 3 = 0$.

(3)

(b) Hence solve the equation for x, in the interval $[0, \pi]$.

(6)
(Total 9 marks)

42. Consider triangle ABC with $\text{B}\hat{\text{A}}\text{C} = 37.8°$, AB = 8.75 and BC = 6.

Find AC.

(Total 7 marks)

43. If x satisfies the equation $\sin\left(x + \frac{\pi}{3}\right) = 2 \sin x \sin\left(\frac{\pi}{3}\right)$, show that $11 \tan x = a + b\sqrt{3}$, where $a, b \in \mathbb{Z}^+$.

(Total 6 marks)

44. Given ΔABC, with lengths shown in the diagram below, find the length of the line segment [CD].

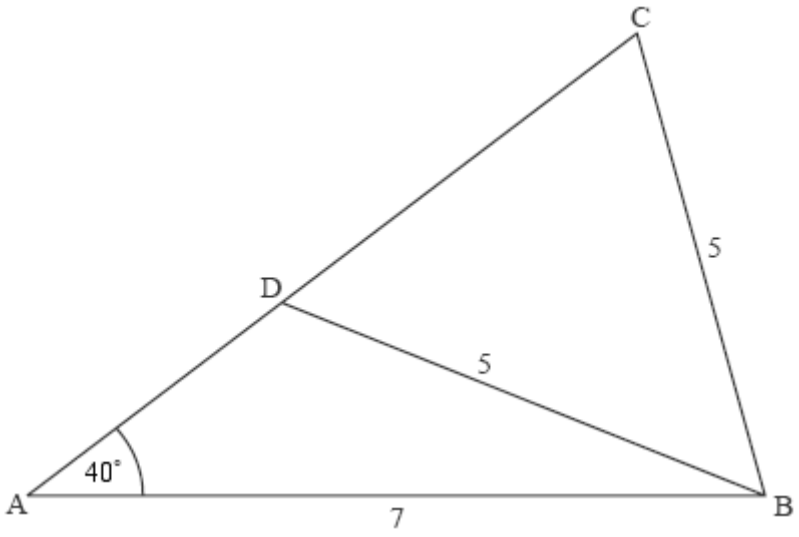


diagram not to scale

(Total 5 marks)