MATH HL2 EXAM PREP – CORE TOPICS – VECTORS (SOLUTIONS)

1. (a) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$ A1

$\overrightarrow{CB} = \boldsymbol{a} + \boldsymbol{b}$ A1

(b) $\overrightarrow{AB} \bullet \overrightarrow{CB} = (\boldsymbol{b} - \boldsymbol{a}) \bullet (\boldsymbol{b} + \boldsymbol{a})$ M1

$= |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2$ A1

$= 0$ since $|\boldsymbol{b}| = |\boldsymbol{a}|$ R1

Note: Only award the A1 and R1 if working indicates that they understand that they are working with vectors.

so $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CB}$ i.e. $A\hat{B}C$ is a right angle AG

[5]

2. (a) $\overrightarrow{AB} = \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$ A1A1

Note: Accept row vectors.

(b) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -4 & -1 & 3 \\ 4 & -3 & 1 \end{vmatrix} = \begin{pmatrix} 8 \\ 16 \\ 16 \end{pmatrix}$ M1A1

normal $\boldsymbol{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ so $\boldsymbol{r} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ (M1)

$x + 2y + 2z = 7$ A1

Note: If attempt to solve by a system of equations:
Award A1 for 3 correct equations, A1 for eliminating a variable and A2 for the correct answer.

(c) $\boldsymbol{r} = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ (or equivalent) A1

$1(5 + \lambda) + 2(3 + 2\lambda) + 2(7 + 2\lambda) = 7$ M1
$9\lambda = -18$
$\lambda = -2$ A1

Note: $\lambda = -\frac{1}{4}$ if $\begin{pmatrix} 8 \\ 16 \\ 16 \end{pmatrix}$ is used.

distance = $2\sqrt{1^2 + 2^2 + 2^2}$ (M1)
$= 6$ A1

(d) (i) area = $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{8^2 + 16^2 + 16^2}$ (M1)
$= 12$ (accept $\frac{1}{2}\sqrt{576}$) A1

(ii) **EITHER**

volume = $\frac{1}{3}$ × area × height (M1)
$= \frac{1}{3} \times 12 \times 6 = 24$ A1

OR

volume = $\frac{1}{6}\left(\overrightarrow{AD} \bullet (\overrightarrow{AB} \times \overrightarrow{AC})\right)$ M1
$= 24$ A1

(e) $\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{8^2+16^2+16^2}$

$\left|\overrightarrow{AC}\times\overrightarrow{AD}\right| = \left\|\begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & -3 & 1 \\ 4 & 1 & 6 \end{matrix}\right\|$ M1

$= \left|-19\boldsymbol{i} - 20\boldsymbol{j} + 16\boldsymbol{k}\right|$ A1

EITHER

$\frac{1}{2}\sqrt{19^2+20^2+16^2} > \frac{1}{2}\sqrt{8^2+16^2+16^2}$ M1

therefore since area of ACD bigger than area ABC implies that
B is closer to opposite face than D R1

OR

correct calculation of second distance as $\frac{144}{\sqrt{19^2+20^2+16^2}}$ A1

which is smaller than 6 R1

Note: Only award final R1 in each case if the calculations are correct.

[19]

3. (a) $\overrightarrow{CB} = \boldsymbol{b} - \boldsymbol{c}$, $\overrightarrow{AC} = \boldsymbol{b} + \boldsymbol{c}$ A1A1

Note: Condone absence of vector notation in (a).

(b) $\overrightarrow{AC}\bullet\overrightarrow{CB} = (\boldsymbol{b}+\boldsymbol{c})\bullet(\boldsymbol{b}-\boldsymbol{c})$ M1

$= |\boldsymbol{b}|^2 - |\boldsymbol{c}|^2$ A1

$= 0$ since $|\boldsymbol{b}| = |\boldsymbol{c}|$ R1

Note: Only award the A1 and R1 if working indicates that they understand that they are working with vectors.

so $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{CB}$ i.e. $\mathrm{A\hat{C}B}$ is a right angle AG

[5]

4. **METHOD 1**

equation of journey of ship S_1

$\boldsymbol{r}_1 = t\begin{pmatrix}10\\20\end{pmatrix}$

equation of journey of speedboat S_2 ,setting off k minutes later

$\boldsymbol{r}_2 = \begin{pmatrix}70\\30\end{pmatrix} + (t-k)\begin{pmatrix}-60\\30\end{pmatrix}$ M1A1A1

Note: Award M1 for perpendicular direction, A1 for speed, A1 for change in parameter (e.g. by using $t - k$ or T, k being the time difference between the departure of the ships).

solve $t\begin{pmatrix}10\\20\end{pmatrix} = \begin{pmatrix}70\\30\end{pmatrix} + (t-k)\begin{pmatrix}-60\\30\end{pmatrix}$ (M1)

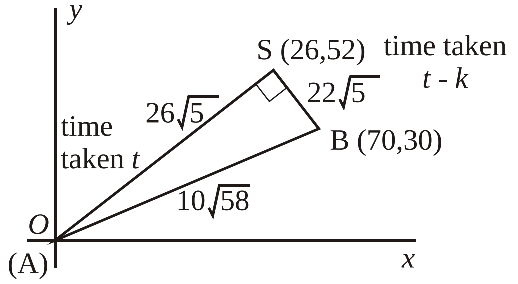
Note: *M* mark is for equating their two expressions.

$10t = 70 - 60t + 60k$
$20t = 30 + 30t - 30k$ M1

Note: *M* mark is for obtaining two equations involving two different parameters.

$7t - 6k = 7$
$-t + 3k = 3$
$k = \frac{28}{15}$ A1
latest time is 11:52 A1

METHOD 2



SB = $22\sqrt{5}$ M1A1

(by perpendicular distance)

SA = $26\sqrt{5}$ M1A1

(by Pythagoras or coordinates)

$t = \frac{26\sqrt{5}}{10\sqrt{5}}$ A1

$t - k = \frac{22\sqrt{5}}{30\sqrt{5}}$ A1

$k = \frac{28}{15}$ leading to latest time 11:52 A1

[7]

5. (a) $\begin{pmatrix} 0 & 2 & 1 \\ -1 & 1 & 3 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ k \end{pmatrix}$

$\begin{vmatrix} 0 & 2 & 1 \\ -1 & 1 & 3 \\ -2 & 1 & 2 \end{vmatrix} = 0 - 2(-2 + 6) + (-1 + 2) = -7$ M1A1

since determinant $\neq 0 \Rightarrow$ unique solution to the system planes R1

intersect in a point AG

Note: For any method, including row reduction, leading to the explicit solution $\left(\frac{6-5k}{7}, \frac{10+k}{7}, \frac{1-2k}{7}\right)$, award M1 for an attempt at a correct method, A1 for two correct coordinates and A1 for a third correct coordinate.

(b) $\begin{vmatrix} a & 2 & 1 \\ -1 & a+1 & 3 \\ -2 & 1 & a+2 \end{vmatrix}$

$= a((a + 1)(a + 2) – 3) – 2(–1(a + 2) + 6) + (–1 + 2(a + 1))$ M1(A1)

planes not meeting in a point $\Rightarrow$ no unique solution i.e. determinant = 0(M1)

$a(a^2 + 3a – 1) + (2a – 8) + (2a + 1) = 0$

$a^3 + 3a^2 + 3a – 7 = 0$ A1

$a = 1$ A1

[5]

(c) $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ -2 & 1 & 3 & k \end{pmatrix}$ $r_1 + r_2$ M1

$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 5 & 5 & 6+k \end{pmatrix}$ $2r_1 + r_3$ (A1)

$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 4+4k \end{pmatrix}$ $4r_3 - 5r_2$ (A1)

for an infinite number of solutions to exist, $4 + 4k = 0 \Rightarrow k = –1$ A1

$x + 2y + z = 3$

$y + z = 1$ M1

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ A1

Note: Accept methods involving elimination.

Note: Accept any equivalent form e.g. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Award A0 if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$ or $\boldsymbol{r} =$ is absent.

[14]

6. (a) $x^3 + 1 = \dfrac{1}{x^3 + 1}$

$(–1.26, –1)$ $(= (-\sqrt[3]{2}, –1))$ A1

(b) $f'(–1.259...) = 4.762...$ $(3 \times 2^{\frac{2}{3}})$ A1

$g'(-1.259...) = -4.762...$ $\quad (-3 \times 2^{\frac{2}{3}})$ A1

required angle $= 2\arctan\left(\frac{1}{4.762...}\right)$ M1

$= 0.414$ (accept 23.7°) A1

Note: Accept alternative methods including finding the obtuse angle first.

[5]

7. (a) $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}, \overrightarrow{SR} = \begin{pmatrix} 0-x \\ 5-y \\ 1-z \end{pmatrix}$ (M1)

point S = (1, 6, –2) A1

(b) $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \quad \overrightarrow{PS} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ A1

$\overrightarrow{PQ} \times \overrightarrow{PS} = \begin{pmatrix} -13 \\ 7 \\ -2 \end{pmatrix}$

$m = -2$ A1

(c) area of parallelogram PQRS $= \left|\overrightarrow{PQ} \times \overrightarrow{PS}\right| = \sqrt{(-13)^2 + 7^2 + (-2)^2}$ M1

$= \sqrt{222} = 14.9$ A1

(d) equation of plane is $-13x + 7y - 2z = d$ M1A1
substituting any of the points given gives $d = 33$
$-13x + 7y - 2z = 33$ A1

(e) equation of line is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -13 \\ 7 \\ -2 \end{pmatrix}$ A1

Note: To get the A1 must have ***r*** = or equivalent.

(f) $169\lambda + 49\lambda + 4\lambda = 33$ M1

$\lambda = \dfrac{33}{222}$ (= 0.149...) A1

closest point is $\left(-\dfrac{143}{74}, \dfrac{77}{74}, -\dfrac{11}{37}\right)$ (= (–1.93, 1.04, –0.297)) A1

(g) angle between planes is the same as the angle between the normals (R1)

$\cos\theta = \dfrac{-13\times1+7\times-2-2\times1}{\sqrt{222}\times\sqrt{6}}$ M1A1

$\theta = 143°$ (accept $\theta = 37.4°$ or 2.49 radians or 0.652 radians) A1

[17]

8. (a) for using normal vectors (M1)

$\begin{pmatrix}1\\2\\-1\end{pmatrix}\bullet\begin{pmatrix}1\\0\\1\end{pmatrix} = 1 - 1 = 0$ M1A1

hence the two planes are perpendicular AG

(b) **METHOD 1**

EITHER

$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\1 & 2 & -1\\1 & 0 & 1\end{vmatrix} = -2\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$ M1A1

OR

if $\begin{pmatrix}a\\b\\c\end{pmatrix}$ is normal to π_3, then

$a + 2b - c = 0$ and $a + c = 0$ M1

a solution is $a = 1$, $b = -1$, $c = -1$ A1

THEN

π_3 has equation $x - y - z = d$ (M1)

as it goes through the origin, $d = 0$ so π_3 has equation $x - y - z = 0$ A1

Note: The final (M1)A1 are independent of previous working.

METHOD 2

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ A1(A1)A1A1

[7]

9. $(\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a} \bullet \boldsymbol{a} + \boldsymbol{b} \bullet \boldsymbol{a} - \boldsymbol{a} \bullet \boldsymbol{b} - \boldsymbol{b} \bullet \boldsymbol{b}$ M1

$= \boldsymbol{a} \bullet \boldsymbol{a} - \boldsymbol{b} \bullet \boldsymbol{b}$ A1

$= |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2 = 0$ since $|\boldsymbol{a}| = |\boldsymbol{b}|$ A1

the **diagonals** are perpendicular R1

Note: Accept geometric proof, awarding M1 for recognizing OACB is a rhombus, R1 for a clear indication that $(\boldsymbol{a} + \boldsymbol{b})$ and $(\boldsymbol{a} - \boldsymbol{b})$ are the diagonals, A1 for stating that diagonals cross at right angles and A1 for "hence dot product is zero".

Accept solutions using components in 2 or 3 dimensions.

[4]

10. (a) $2y + 8x = 4$ M1

$-3x + 2y = -7$ A1

$2x + 6 - 2x = 6$

Note: Award M1 for attempt at components, A1 for two correct equations. No penalty for not checking the third equation.

solving : $x = 1$, $y = -2$ A1

(b) $|\boldsymbol{a} + 2\boldsymbol{b}| = \left|\begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}\right|$

$= \left|\begin{pmatrix} 4 \\ -7 \\ 6 \end{pmatrix}\right|$

$\Rightarrow |\boldsymbol{a} + 2\boldsymbol{b}| = \sqrt{4^2 + (-7)^2 + 6^2}$ (M1)

$= \sqrt{101}$ A1

[5]

11. (a) (i) use of $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ (M1)

$\boldsymbol{a} \bullet \boldsymbol{b} = -1$ (A1)

$|\boldsymbol{a}| = 7$, $|\boldsymbol{b}| = 5$ (A1)

$\cos\theta = -\frac{1}{35}$ A1

(ii) the required cross product is

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 6 & 3 & 2 \\ 0 & -3 & 4 \end{vmatrix} = 18\boldsymbol{i} - 24\boldsymbol{j} - 18\boldsymbol{k}$$ M1A1

(iii) using $\boldsymbol{r} \bullet \boldsymbol{n} = \boldsymbol{p} \bullet \boldsymbol{n}$ the equation of the plane is (M1)

$18x - 24y - 18z = 12$ $(3x - 4y - 3z = 2)$ A1

(iv) recognizing that $z = 0$ (M1)

x-intercept $= \frac{2}{3}$, y-intercept $= -\frac{1}{2}$ (A1)

area $= \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$ A1

(b) (i) $\boldsymbol{p} \bullet \boldsymbol{p} = |\boldsymbol{p}||\boldsymbol{p}|\cos 0$ M1A1

$= |\boldsymbol{p}|^2$ AG

(ii) consider the LHS, and use of result from part (i)

$|\boldsymbol{p} + \boldsymbol{q}|^2 = (\boldsymbol{p} + \boldsymbol{q}) \bullet (\boldsymbol{p} + \boldsymbol{q})$ M1

$= \boldsymbol{p} \bullet \boldsymbol{p} + \boldsymbol{p} \bullet \boldsymbol{q} + \boldsymbol{q} \bullet \boldsymbol{p} + \boldsymbol{q} \bullet \boldsymbol{q}$ (A1)

$= \boldsymbol{p} \bullet \boldsymbol{p} + 2\boldsymbol{p} \bullet \boldsymbol{q} + \boldsymbol{q} \bullet \boldsymbol{q}$ A1

$= |\boldsymbol{p}|^2 + 2\boldsymbol{p} \bullet \boldsymbol{q} + |\boldsymbol{q}|^2$ AG

(iii) **EITHER**

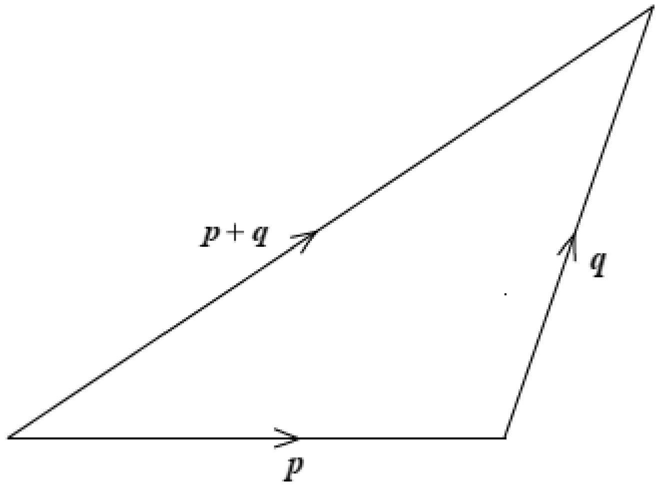
use of $\boldsymbol{p} \cdot \boldsymbol{q} \le |\boldsymbol{p}|\,|\boldsymbol{q}|$ M1

so $0 \le |\boldsymbol{p} + \boldsymbol{q}|^2 = |\boldsymbol{p}|^2 + 2\boldsymbol{p} \cdot \boldsymbol{q} + |\boldsymbol{q}|^2 \le |\boldsymbol{p}|^2 + 2|\boldsymbol{p}|\,|\boldsymbol{q}| + |\boldsymbol{q}|^2$ A1

take square root (of these positive quantities) to establish A1

$|\boldsymbol{p} + \boldsymbol{q}| \le |\boldsymbol{p}| + |\boldsymbol{q}|$ AG

OR



M1M1

Note: Award M1 for correct diagram and M1 for correct labelling of vectors including arrows.

since the sum of any two sides of a triangle is greater than the third side,

$|\boldsymbol{p}| + |\boldsymbol{q}| > |\boldsymbol{p} + \boldsymbol{q}|$ A1

when p and q are collinear $|\boldsymbol{p}| + |\boldsymbol{q}| = |\boldsymbol{p} + \boldsymbol{q}|$

$\Rightarrow |\boldsymbol{p} + \boldsymbol{q}| \le |\boldsymbol{p}| + |\boldsymbol{q}|$ AG

[19]

12. **EITHER**

using row reduction (or attempting to eliminate a variable) M1

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 3 & 1 & 2 & -2 \\ -1 & 2 & a & b \end{array}\right) \begin{array}{l} \\ \to 2R2-3R1 \\ \to 2R3+R1 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 5 & -5 & -10 \\ 0 & 3 & 2a+3 & 2b+2 \end{array}\right) \to R2/5$$ A1

Note: For an algebraic solution award A1 for **two** correct equations in two variables.

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & 2a+3 & 2b+2 \end{array}\right) \to R3-3R2$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2a+6 & 2b+8 \end{array}\right)$$

Note: Accept alternative correct row reductions.

recognition of the need for 4 zeroes M1
so for multiple solutions $a = -3$ and $b = -4$ A1A1

OR

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 2 \\ -1 & 2 & a \end{vmatrix} = 0$$ M1

$\Rightarrow\ 2(a-4) + (3a+2) + 3(6+1) = 0$
$\Rightarrow\ 5a + 15 = 0$
$\Rightarrow\ a = -3$ A1

$$\begin{vmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ -1 & 2 & b \end{vmatrix} = 0$$ M1

$\Rightarrow\ 2(b+4) + (3b-2) + 2(6+1) = 0$ A1
$\Rightarrow\ 5b + 20 = 0$
$\Rightarrow\ b = -4$ A1

[5]

13. (a) **EITHER**

normal to plane given by

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & 2 \\ 6 & -3 & 2 \end{vmatrix}$$ M1A1

$= 12\boldsymbol{i} + 8\boldsymbol{j} - 24\boldsymbol{k}$ A1

equation of π is $3x + 2y - 6z = d$ (M1)

as goes through $(-2, 3, -2)$ so $d = 12$ M1A1

$\pi : 3x + 2y - 6z = 12$ AG

OR

$x = -2 + 2\lambda + 6\mu$
$y = 3 + 3\lambda - 3\mu$
$z = -2 + 2\lambda + 2\mu$

eliminating μ
$x + 2y = 4 + 8\lambda$
$2y + 3z = 12\lambda$ M1A1A1
eliminating λ
$3(x + 2y) - 2(2y + 3z) = 12$ M1A1A1
$\pi : 3x + 2y - 6z = 12$ AG

(b) therefore A(4, 0, 0), B(0, 6, 0) and C(0, 0, –2) A1A1A1

Note: Award A1A1A0 if position vectors given instead of coordinates.

(c) area of base OAB = $\frac{1}{2} \times 4 \times 6 = 12$ M1

$V = \frac{1}{3} \times 12 \times 2 = 8$ M1A1

(d) $\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3 = 7 \times 1 \times \cos \phi$ M1A1

$\phi = \arccos \frac{3}{7}$

so $\theta = 90 - \arccos \frac{3}{7} = 25.4°$ (accept 0.443 radians) M1A1

(e) $d = 4 \sin \theta = \frac{12}{7}$ (= 1.71) (M1)A1

(f) $8 = \frac{1}{3} \times \frac{12}{7} \times \text{area} \Rightarrow$ area = 14 M1A1

Note: If answer to part (f) is found in an earlier part, award M1A1, regardless of the fact that it has not come from their answers to part (c) and part (e).

[20]

14. (a) use GDC or manual method to find a, b and c (M1)
obtain $a = 2$, $b = -1$, $c = 3$ (in any identifiable form) A1

(b) use GDC or manual method to solve second set of equations (M1)
obtain $x = \frac{4-11t}{2}$; $y = \frac{-7t}{2}$, $z = t$ (or equivalent) (A1)

$\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5.5 \\ -3.5 \\ 1 \end{pmatrix}$ (accept equivalent vector forms) M1A1

Note: Final A1 requires $\boldsymbol{r}$ = or equivalent.

[6]

15. (a) $\boldsymbol{a} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} \perp$ tothe plane $\boldsymbol{e} = \begin{pmatrix} -2 \\ 1 \\ k \end{pmatrix}$ is parallel to the line (A1)(A1)

Note: Award A1 for each correct vector written down, even if not identified.

line $\perp$ plane $\Rightarrow$ $\boldsymbol{e}$ parallel to $\boldsymbol{a}$

since $\begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ k \end{pmatrix} \Rightarrow k = \frac{1}{2}$ (M1)A1

(b) $4(3-2\lambda)-2\lambda-\left(-1+\frac{1}{2}\lambda\right)=1$ (M1)(A1)

Note: FT their value of *k* as far as possible.

$\lambda=\frac{8}{7}$ A1

$\mathrm{P}\left(\frac{5}{7},\frac{8}{7},-\frac{3}{7}\right)$ A1

[8]

16. (a) $\cos\theta=\frac{\boldsymbol{ab}}{|\boldsymbol{a}||\boldsymbol{b}|}=\frac{\sin 2\alpha\cos\alpha+\sin\alpha\cos 2\alpha-1}{\sqrt{2}\times\sqrt{2}}\left(=\frac{\sin 3\alpha-1}{2}\right)$ M1A1

(b) $\boldsymbol{a}\perp\boldsymbol{b}\Rightarrow\cos\theta=0$ M1

$\sin 2\alpha\cos\alpha+\sin\alpha\cos 2\alpha-1=0$

$\alpha=0.524\left(=\frac{\pi}{6}\right)$ A1

(c) **METHOD 1**

$$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \sin 2\alpha & -\cos 2\alpha & 1\\ \cos\alpha & -\sin\alpha & -1\end{vmatrix}$$ (M1)

assuming $\alpha=\frac{7\pi}{6}$

Note: Allow substitution at any stage.

$$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1\\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & -1\end{vmatrix}$$ A1

$=\boldsymbol{i}\left(\frac{1}{2}-\frac{1}{2}\right)-\boldsymbol{j}\left(-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right)+\boldsymbol{k}\left(\frac{\sqrt{3}}{2}\times\frac{1}{2}-\frac{1}{2}\times\frac{\sqrt{3}}{2}\right)$

$=0$ A1

a and ***b*** are parallel R1

Note: Accept decimal equivalents.

METHOD 2

from (a) $\cos\theta = -1$ (and $\sin\theta = 0$) M1A1
$\boldsymbol{a} \times \boldsymbol{b} = 0$ A1
$\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel R1

[8]

17. (a) $\overrightarrow{OM} = \begin{pmatrix}1\\2\\2\end{pmatrix}, \overrightarrow{ON} = \begin{pmatrix}0\\1\\2\end{pmatrix}$ and $\overrightarrow{OP} = \begin{pmatrix}0\\2\\1\end{pmatrix}$ A1A1A1

(b) $\overrightarrow{MP} = \begin{pmatrix}-1\\0\\-1\end{pmatrix}$ and $\overrightarrow{MN} = \begin{pmatrix}-1\\-1\\0\end{pmatrix}$, A1A1

$\overrightarrow{MP} \times \overrightarrow{MN} = \begin{pmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\-1 & 0 & -1\\-1 & -1 & 0\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$ (M1)A1

(c) (i) area of MNP $= \frac{1}{2}\left|\overrightarrow{MP} \times \overrightarrow{MN}\right|$ M1

$= \frac{1}{2}\left|\begin{pmatrix}-1\\1\\1\end{pmatrix}\right|$

$= \frac{\sqrt{3}}{2}$ A1

(ii) $\overrightarrow{OA} = \begin{pmatrix}2\\0\\0\end{pmatrix}, \overrightarrow{OG} = \begin{pmatrix}0\\2\\2\end{pmatrix}$

$\overrightarrow{AG} = \begin{pmatrix}-2\\2\\2\end{pmatrix}$ A1

since $\overrightarrow{AG} = 2(\overrightarrow{MP} \times \overrightarrow{MN})$ AG is perpendicular to MNP R1

(iii) $\boldsymbol{r} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ M1A1

$\boldsymbol{r} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$ (accept $-x + y + z = 3$) A1

(d) $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$ A1

$\begin{pmatrix} 2-2\lambda \\ 2\lambda \\ 2\lambda \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$ M1A1

$-2 + 2\lambda + 2\lambda + 2\lambda = 3$

$\lambda = \frac{5}{6}$ A1

$\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \frac{5}{6} \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$ M1

coordinates of point $\left(\frac{1}{3}, \frac{5}{3}, \frac{5}{3}\right)$ A1

[20]

18. METHOD 1

for finding two of the following three vectors (or their negatives)

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$$ (A1)(A1)

and calculating

EITHER

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 2 & -1 \\ -2 & 2 & -2 \end{vmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$$ M1A1

area $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ M1

OR

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -2 & 1 \\ -2 & 0 & -1 \end{vmatrix} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$$ M1A1

area $\Delta ABC = \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|$ M1

OR

$$\overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -2 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$$ M1A1

area $\Delta ABC = \frac{1}{2}\left|\overrightarrow{CA} \times \overrightarrow{CB}\right|$ M1

THEN

area $\Delta ABC = \frac{\sqrt{24}}{2}$ A1

$= \sqrt{6}$ AG N0

METHOD 2

for finding two of the following three vectors (or their negatives)

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ (A1)(A1)

EITHER

$\cos A = \dfrac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$ M1

$= \dfrac{6}{\sqrt{5}\sqrt{12}} = \dfrac{6}{\sqrt{60}} \left(\text{or } \dfrac{3}{\sqrt{15}}\right)$

$\sin A = \sqrt{\dfrac{2}{5}}$ A1

area $\Delta ABC = \dfrac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}| \sin A$ M1

$= \dfrac{1}{2}\sqrt{5}\sqrt{12}\sqrt{\dfrac{2}{5}}$

$= \dfrac{1}{2}\sqrt{24}$ A1

$= \sqrt{6}$ AG N0

OR

$\cos B = \dfrac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$ M1

$= -\dfrac{1}{\sqrt{5}\sqrt{5}} = -\dfrac{1}{5}$

$\sin B = \sqrt{\dfrac{24}{25}} \left(\text{or } \dfrac{\sqrt{24}}{5}\right)$ A1

area $\Delta ABC = \dfrac{1}{2}|\overrightarrow{BA}||\overrightarrow{BC}| \sin B$ M1

$= \dfrac{1}{2}\sqrt{5}\sqrt{5}\sqrt{\dfrac{24}{25}}$

$= \dfrac{1}{2}\sqrt{24}$ A1

$= \sqrt{6}$ AG N0

OR

$\cos C = \dfrac{\overrightarrow{CA} \bullet \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}$ M1

$= \dfrac{6}{\sqrt{12}\sqrt{5}} = \dfrac{6}{\sqrt{60}} \left(\text{or } \dfrac{3}{\sqrt{15}} \right)$

$\sin C = \sqrt{\dfrac{2}{5}}$ A1

area ΔABC $= \dfrac{1}{2}|\overrightarrow{CA}||\overrightarrow{CB}| \sin C$ M1

$= \dfrac{1}{2}\sqrt{12}\sqrt{5}\sqrt{\dfrac{2}{5}}$

$= \dfrac{1}{2}\sqrt{24}$ A1

$= \sqrt{6}$ AG N0

METHOD 3

for finding two of the following three vectors (or their negatives)

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ (A1)(A1)

AB $= \sqrt{5} = c$, AC $= \sqrt{12} = 2\sqrt{3} = b$, BC $= \sqrt{5} = a$ M1A1

$s = \dfrac{\sqrt{5} + 2\sqrt{3} + \sqrt{5}}{2} = \sqrt{3} + \sqrt{5}$ M1

area ΔABC $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{(\sqrt{3}+\sqrt{5})(\sqrt{3})(\sqrt{5}-\sqrt{3})(\sqrt{3})}$

$= \sqrt{3(5-3)}$ A1

$= \sqrt{6}$ AG N0

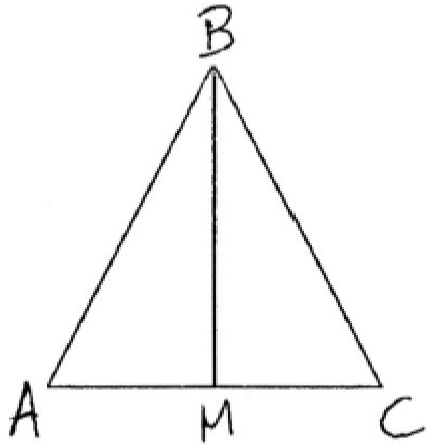
METHOD 4

for finding two of the following three vectors (or their negatives)

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ (A1)(A1)

AB = BC = $\sqrt{5}$ and AC = $\sqrt{12} = 2\sqrt{3}$ M1A1

ΔABC is isosceles



let M be the midpoint of [AC], the height BM = $\sqrt{5-3} = \sqrt{2}$ M1

area ΔABC = $\frac{2\sqrt{3} \times \sqrt{2}}{2}$ A1

= $\sqrt{6}$ AG N0

[6]

19. (a) identifies a direction vector e.g. $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ or $\overrightarrow{BA} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$ A1

identifies the point (1, –1, 2) A1

line l_1: $\frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{1}$ AG

(b) $\boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$1 + 2\lambda = 1 + \mu$, $-1 + \lambda = 2 + 2\mu$, $2 + \lambda = 3 + \mu$ (M1)

equating two of the three equations gives $\lambda = -1$ and $\mu = -2$ A1A1

check in the third equation

satisfies third equation therefore the lines intersect R1

therefore coordinates of intersection are (–1, –2, 1) A1

(c) $\boldsymbol{d}_1 = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{d}_2 = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ A1

$\boldsymbol{d}_1 \times \boldsymbol{d}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ M1A1

Note: Accept scalar multiples of above vectors.

(d) equation of plane is $-x - y + 3z = k$ M1A1
contains (1, 2, 3) (or (–1, –2, 1) or (1, –1, 2)) $\therefore$ $k = -1 - 2 + 3 \times 3 = 6$ A1
$-x - y + 3z = 6$ AG

(e) direction vector of the perpendicular line is $\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ (M1)

$\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} + m \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ A1

Note: Award A0 if $\boldsymbol{r}$ omitted.

(f) (i) find point where line meets plane
$-(3 - m) - (1 - m) + 3(-4 + 3m) = 6$ M1
$m = 2$ A1
point of intersection is (1, –1, 2) A1

(ii) for T′, $m = 4$ (M1)
so T′ = (–1, –3, 8) A1

(iii) $\overrightarrow{\text{TT}'} = \sqrt{(3+1)^2 + (1+3)^2 + (-4-8)^2}$ (M1)
$= \sqrt{176}$ $(= 4\sqrt{11})$ A1

[22]

20. consider a vector parallel to each line,

e.g. $\boldsymbol{u} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$ A1A1

let θ be the angle between the lines

$\cos\theta = \dfrac{|\boldsymbol{u} \bullet \boldsymbol{v}|}{|\boldsymbol{u}||\boldsymbol{v}|} = \dfrac{|12-6+1|}{\sqrt{21}\sqrt{19}}$ M1A1

$= \dfrac{7}{\sqrt{21}\sqrt{19}} = 0.350...$ (A1)

so $0 = 69.5°$ $\left(\text{or } 1.21 \text{ rad or } \arccos\left(\dfrac{7}{\sqrt{21}\sqrt{19}}\right)\right)$ A1 N4

Note: Allow FT from incorrect reasonable vectors.

[6]

21. (a) let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & 4 \end{pmatrix}$, $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$ (M1)

point of intersection is $\left(\dfrac{11}{12}, \dfrac{7}{12}, \dfrac{1}{4}\right)$ (or (0.917, 0.583, 0.25)) A1

(b) **METHOD 1**

(i) $\det\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & a \end{pmatrix} = 0$ M1

$-3a + 24 = 0$ (A1)

$a = 8$ A1 N1

(ii) consider the augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & 8 & 5 \end{array}\right)$ M1

use row reduction to obtain $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{array}\right)$ or $\left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

(or equivalent) A1

any valid reason R1

(e.g. as the last row is not all zeros, the planes do not meet) N0

METHOD 2

use of row reduction (or equivalent manipulation of equations) M1

e.g. $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & 5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & a-10 & -5 \end{array}\right)$ A1A1

Note: Award an A1 for each correctly reduced row.

(i) $a - 10 = -2 \Rightarrow a = 8$ M1A1 N1

(ii) when $a = 8$, row 3 $\neq$ 2 × row 2 R1 N0

[8]

22. (a) $\overrightarrow{OP} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ (M1)

the coordinates of P are (1, 2, –1) A1

(b) **EITHER**

$x = 1 + t$, $y = 2 - 2t$, $z = 3t - 1$ M1

$x - 1 = t$, $\frac{y-2}{-2} = t$, $\frac{z+1}{3} = t$ A1

$x - 1 = \frac{y-2}{-2} = \frac{z+1}{3}$ AG N0

OR

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ M1A1

$x - 1 = \frac{y-2}{-2} = \frac{z+1}{3}$ AG

(c) (i) $2(1 + t) + (2 - 2t) + (3t - 1) = 6 \Rightarrow t = 1$ M1A1 N1

(ii) coordinates are (2, 0, 2) A1

Note: Award A0 for position vector.

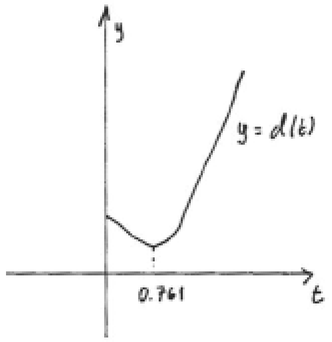
(iii) distance travelled is the distance between the two points (M1)

$\sqrt{(2-1)^2 + (0-2)^2 + (2+1)^2} = \sqrt{14}$ (= 3.74) (M1)A1

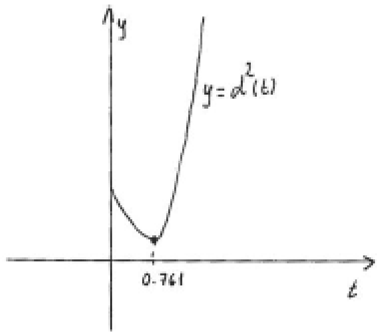
(d) (i) distance from Q to the origin is given by

$d(t) = \sqrt{t^4 + (1-t)^2 + (1-t^2)^2}$ (or equivalent) M1A1

e.g. for labelled sketch of graph of d or d^2 (M1)(A1)



or



the minimum value is obtained for $t = 0.761$ A1 N3

(ii) the coordinates are (0.579, 0.239, 0.421) A1

Note: Accept answers given as a position vector.

(e) (i) $\boldsymbol{a} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ (M1)A1

substituting in the equation $\boldsymbol{a} - \boldsymbol{b} = k(\boldsymbol{b} - \boldsymbol{c})$, we have (M1)

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = k\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}\right) \Leftrightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = k\begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$ A1

$\Rightarrow k = 1$ and $k = \frac{1}{3}$ which is impossible

so there is no solution for k R1

(ii) $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are not parallel R2

(hence A, B, and C cannot be collinear)

Note: Only accept answers that follow from part (i).

[23]

23. direction vector for line = $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or any multiple A1

$\begin{pmatrix} 2\sin\theta \\ 1-\sin\theta \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$ M1

$2 \sin \theta - 1 + \sin \theta = 0$ A1

Note: Allow FT on candidate's direction vector just for line above only.

$3 \sin \theta = 1$

$\sin \theta = \frac{1}{3}$ A1

$\theta = 0.340$ or $19.5°$ A1

Note: A coordinate geometry method using perpendicular gradients is acceptable.

[5]

24. EITHER

l goes through the point (1, 3, 6), and the plane contains A(4, –2, 5)
the vector containing these two points is on the plane, i.e.

$\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$ (M1)A1

$\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & -1 \\ -3 & 5 & 1 \end{vmatrix} = 7\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ M1A1

$\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} = 25$ (M1)

hence, Cartesian equation of the plane is $7x + 4y + z = 25$ A1

OR

finding a third point M1
e.g. (0, 5, 5) A1
three points are (1, 3, 6), (4, –2, 5), (0, 5, 5)
equation is $ax + by + cz = 1$
system of equations
$a + 3b + 6c = 1$ M1
$4a - 2b + 5c = 1$
$5b + 5c = 1$
$a = \frac{7}{25}, b = \frac{4}{25}, c = \frac{1}{25}$, from GDC M1A1
so $\frac{7}{25}x + \frac{4}{25}y + \frac{1}{25}z = 1$ A1
or $7x + 4y + z = 25$

[6]

25. (a) on l_1 A$(-3 + 3\lambda, -4 + 2\lambda, 6 - 2\lambda)$ A1

on l_2 $l_2 : \boldsymbol{r} = \begin{pmatrix} 4 \\ -7 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$ (M1)

$\Rightarrow$ B$(4 - 3\mu, -7 + 4\mu, -3 - \mu)$ A1

$\overrightarrow{\text{BA}} = \boldsymbol{a} - \boldsymbol{b} = \begin{pmatrix} 3\lambda + 3\mu - 7 \\ 2\lambda - 4\mu + 3 \\ -2\lambda + \mu + 9 \end{pmatrix}$ (M1)A1

EITHER

$\text{BA} \perp l_1 \Rightarrow \text{BA} \bullet \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = 0$

$\Rightarrow\ 3(3\lambda + 3\mu - 7) + 2(2\lambda - 4\mu + 3) - 2(-2\lambda + \mu + 9) = 0$ M1

$\Rightarrow 17\lambda - \mu = 33$ A1

$\text{BA} \perp l_2 \Rightarrow \text{BA} \bullet \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = 0$

$\Rightarrow\ -3(3\lambda + 3\mu - 7) + 4(2\lambda - 4\mu + 3) - 2(-2\lambda + \mu + 9) = 0$ M1

$\Rightarrow\ \lambda - 26\mu = -24$ A1

solving both equations above simultaneously gives

$\lambda = 2$; $\mu = 1$ $\Rightarrow$ A(3, 0, 2), B(1, –3, –4) A1A1A1A1

OR

$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & -2 \\ -3 & 4 & -1 \end{vmatrix} = 6\boldsymbol{i} + 9\boldsymbol{j} + 18\boldsymbol{k}$ M1A1

so $\overrightarrow{\text{AB}} = p\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3\lambda + 3\mu - 7 \\ 2\lambda - 4\mu + 3 \\ -2\lambda + \mu + 9 \end{pmatrix}$ M1A1

$3\lambda + 3\mu - 2p = 7$

$2\lambda - 4\mu - 3p = -3$

$-2\lambda + \mu - 6p = -9$

$\lambda = 2, \mu = 1, p = 1$ A1A1

A(–3 + 6, –4 + 4, 6 – 4) = (3, 0, 2) A1

B(4 – 3, –7 + 4, –3 – 1) = (1, –3, –4) A1

(b) $\text{AB} = \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix}$ (A1)

$|\text{AB}| = \sqrt{(-2)^2 + (-3)^2 + (-6)^2} = \sqrt{49} = 7$ M1A1

(c) from (b) $2\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k}$ is normal to both lines

l_1 goes through (–3, –4, 6) $\Rightarrow \begin{pmatrix} -3 \\ -4 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = 18$ M1A1

hence, the Cartesian equation of the plane through l_1, but not l_2,
is $2x + 3y + 6z = 18$ A1

[19]

26. (a) (i) **METHOD 1**

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ (A1)

$\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ (A1)

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix}$ M1

$= \boldsymbol{i}\,(-1 + 1) - \boldsymbol{j}(0 - 2) + \boldsymbol{k}\,(0 - 2)$ (A1)

$= 2\boldsymbol{j} - 2\boldsymbol{k}$ A1

Area of triangle ABC = $\frac{1}{2}\left|2\boldsymbol{j} - 2\boldsymbol{k}\right| = \frac{1}{2}\sqrt{8}\left(= \sqrt{2}\right)$ sq. units M1A1

Note: Allow FT on final A1.

METHOD 2

$|\text{AB}| = \sqrt{2}, |\text{BC}| = \sqrt{12}, |\text{AC}| = \sqrt{6}$ A1A1A1

Using cosine rule, e.g. on $\hat{C}$ M1

$\cos C = \dfrac{6+12-2}{2\sqrt{72}} = \dfrac{2\sqrt{2}}{3}$ A1

$\therefore \text{Area } \Delta\text{ABC} = \dfrac{1}{2}ab\sin C$ M1

$= \dfrac{1}{2}\sqrt{12}\sqrt{6}\sin\left(\arccos\dfrac{2\sqrt{2}}{3}\right)$

$= 3\sqrt{2}\sin\left(\arccos\dfrac{2\sqrt{2}}{3}\right)\ \left(=\sqrt{2}\right)$ A1

Note: Allow FT on final A1.

(ii) AB = $\sqrt{2}$ A1

$\sqrt{2} = \dfrac{1}{2}\text{AB}\times h = \dfrac{1}{2}\sqrt{2}\times h, h$ equals the shortest distance (M1)

$\Rightarrow h = 2$ A1

(iii) **METHOD 1**

π has form $\boldsymbol{r} \bullet \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = d$ (M1)

Since (1, 1, 2) is on the plane

$d = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 2 - 4 = -2$ M1A1

Hence $\boldsymbol{r} \bullet \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = -2$

$2y - 2z = -2$ (or $y - z = -1$) A1

METHOD 2

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ (M1)

$x = 1 + 2\mu$ (i)

$y = 1 + \lambda - \mu$ (ii)

$z = 2 + \lambda - \mu$ (iii) A1

Note: Award A1 for all three correct, A0 otherwise.

From (i) $\mu = \frac{x-1}{2}$

substitute in (ii) $y = 1 + \lambda - \left(\frac{x-1}{2}\right)$

$\Rightarrow \lambda = y - 1 + \left(\frac{x-1}{2}\right)$

substitute λ and μ in (iii) M1

$\Rightarrow z = 2 + y - 1 + \left(\frac{x-1}{2}\right) - \left(\frac{x-1}{2}\right)$

$\Rightarrow y - z = -1$ A1

(b) (i) The equation of OD is

$\boldsymbol{r} = \lambda \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, \left(\text{or } \boldsymbol{r} = \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right)$ M1

This meets π where

$2\lambda + 2\lambda = -1$ (M1)

$\lambda = -\frac{1}{4}$ A1

Coordinates of D are $\left(0, -\frac{1}{2}, \frac{1}{2}\right)$ A1

(ii) $\left|\overrightarrow{\text{OD}}\right| = \sqrt{0 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$ (M1)A1

[20]

27. METHOD 1

Use of $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ (M1)

$|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ (A1)

Note: Only one of the first two marks can be implied.

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2(1 - \cos^2\theta)$ A1

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ (A1)

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (|\boldsymbol{a}||\boldsymbol{b}|\cos\theta)^2$ (A1)

Note: Only one of the above two A1 marks can be implied.

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a} \bullet \boldsymbol{b})^2$ A1

Hence LHS = RHS AG N0

METHOD 2

Use of $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ (M1)

$|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a} \bullet \boldsymbol{b})^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (|\boldsymbol{a}||\boldsymbol{b}|\cos\theta)^2$ (A1)

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ (A1)

Note: Only one of the above two A1 marks can be implied.

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2(1 - \cos^2\theta)$ A1

$= |\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ A1

$= |\boldsymbol{a} \times \boldsymbol{b}|^2$ A1

Hence LHS = RHS AG N0

Notes: Candidates who independently correctlysimplify both sides
and show that LHS = RHS should be awarded full marks.
If the candidate starts off with expression that they are trying to
prove and concludes that $\sin^2\theta = (1 - \cos^2\theta)$ award M1A1A1A1A0A0.
If the candidate uses two general 3D vectors and explicitly finds
the expressions correctly award full marks. Use of 2D vectors
gains a maximum of 2 marks.
If two specific vectors are used no marks are gained.

[6]

28. (a) Use of $\cos\theta = \dfrac{\vec{OA} \bullet \vec{AB}}{\left|\vec{OA}\right|\left|\vec{AB}\right|}$ (M1)

$\vec{AB} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ A1

$\left|\vec{AB}\right| = \sqrt{3}$ **and** $\left|\vec{OA}\right| = 3\sqrt{2}$ A1

$\vec{OA} \bullet \vec{AB} = 6$ A1

substituting gives $\cos\theta = \dfrac{2}{\sqrt{6}} \left(= \dfrac{\sqrt{6}}{3}\right)$ or equivalent M1 N1

(b) L_1: $\boldsymbol{r} = \vec{OA} + s\vec{AB}$ or equivalent (M1)

L_1: $\boldsymbol{r} = \boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$ or equivalent A1

Note: Award (M1)A0 for omitting "**r** =" in the final answer.

(c) Equating components and forming equations involving s and t (M1)
$1 + s = 2 + 2t$, $-1 - s = 4 + t$, $4 + s = 7 + 3t$
Having two of the above three equations A1A1
Attempting to solve for s or t (M1)
Finding either $s = -3$ or $t = -2$ A1
Explicitly showing that these values satisfy the third equation R1
Point of intersection is $(-2, 2, 1)$ A1 N1

Note: Position vector is not acceptable for final A1.

(d) **METHOD 1**

$\boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}$ (A1)

$x = 1 + 2\lambda - 3\mu$, $y = -1 + \lambda + 3\mu$ and $z = 4 + 3\lambda - 3\mu$ M1A1
Elimination of the parameters M1
$x + y = 3\lambda$ so $4(x + y) = 12\lambda$ and $y + z = 4\lambda + 3$
so $3(y + z) = 12\lambda + 9$

$3(y + z) = 4(x + y) + 9$ A1

Cartesian equation of plane is $4x + y - 3z = -9$ (or equivalent) A1 N1

METHOD 2

EITHER

The point (2, 4, 7) lies on the plane.

The vector joining (2, 4, 7) and (1, – 1, 4) and $2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ are parallel to the plane. So they are perpendicular to the normal to the plane.

$(\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}) - (2\boldsymbol{i} + 4\boldsymbol{j} + 7\boldsymbol{k}) = -\boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{k}$ (A1)

$\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -5 & -3 \\ 2 & 1 & 3 \end{vmatrix}$ M1

$= -12\boldsymbol{i} - 3\boldsymbol{j} + 9\boldsymbol{k}$ or equivalent parallel vector A1

OR

L_1 and L_2 intersect at D (–2, 2,1)

$\overrightarrow{\text{AD}} = (-2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) - (\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}) = -3\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k}$ (A1)

$\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 3 \\ -3 & 3 & -3 \end{vmatrix}$ M1

$= -12\boldsymbol{i} - 3\boldsymbol{j} + 9\boldsymbol{k}$ or equivalent parallel vector A1

THEN

$\boldsymbol{r} \bullet \boldsymbol{n} = (\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}) \bullet (-12\boldsymbol{i} - 3\boldsymbol{j} + 9\boldsymbol{k})$ M1

$= 27$ A1

Cartesian equation of plane is $4x + y - 3z = -9$ (or equivalent) A1 N1

[20]

29. The normal vector to the plane is $\begin{pmatrix}1\\3\\2\end{pmatrix}$. (A1)

EITHER

θ is the angle between the line and the normal to the plane.

$$\cos\theta=\frac{\begin{pmatrix}4\\1\\-2\end{pmatrix}\bullet\begin{pmatrix}1\\3\\2\end{pmatrix}}{\sqrt{14}\sqrt{21}}=\frac{3}{\sqrt{14}\sqrt{21}}\quad\left(=\frac{3}{7\sqrt{6}}\right)$$ (M1)A1A1

$\Rightarrow \theta = 79.9°$ (= 1.394 ...) A1

The required angle is 10.1° (= 0.176) A1

OR

ϕ is the angle between the line and the plane.

$$\sin\phi=\frac{\begin{pmatrix}4\\1\\-2\end{pmatrix}\bullet\begin{pmatrix}1\\3\\2\end{pmatrix}}{\sqrt{14}\sqrt{21}}=\frac{3}{\sqrt{14}\sqrt{21}}$$ (M1)A1A1

$\phi = 10.1°$ (= 0.176) A2

[6]

30. **METHOD 1**

(from GDC)

$$\left(\begin{array}{ccc|c}1 & 0 & \frac{1}{6} & -\frac{1}{12}\\ 0 & 1 & -\frac{2}{3} & -\frac{1}{6}\\ 0 & 0 & 0 & 0\end{array}\right)$$ (M1)

$$x+\frac{1}{6}\lambda=-\frac{1}{12}$$ A1

$$y-\frac{2}{3}\lambda=-\frac{1}{6}$$ A1

$$\boldsymbol{r}=\left(-\frac{1}{12}\boldsymbol{i}-\frac{1}{6}\boldsymbol{j}\right)+\lambda\left(-\frac{1}{6}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}+\boldsymbol{k}\right)$$ A1A1A1 N3

METHOD 2

(Elimination method either for equations or row reduction of matrix)

Eliminating one of the variables	M1A1	
Finding a point on the line	(M1)A1	
Finding the direction of the line	M1	
The **vector** equation of the line	A1	N3

[6]

31.	$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$	
	$\overrightarrow{CA} = \boldsymbol{a} - \boldsymbol{c}$	
	$\Rightarrow \boldsymbol{a} \bullet (\boldsymbol{c} - \boldsymbol{b}) = 0$	M1
	and $\boldsymbol{b} \bullet (\boldsymbol{a} - \boldsymbol{c}) = 0$	M1
	$\Rightarrow \boldsymbol{a} \bullet \boldsymbol{c} = \boldsymbol{a} \bullet \boldsymbol{b}$	A1
	and $\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{b} \bullet \boldsymbol{c}$	A1
	$\Rightarrow \boldsymbol{a} \bullet \boldsymbol{c} = \boldsymbol{b} \bullet \boldsymbol{c}$	M1
	$\Rightarrow \boldsymbol{b} \bullet \boldsymbol{c} - \boldsymbol{a} \bullet \boldsymbol{c} = 0$	
	$\boldsymbol{c} \bullet (\boldsymbol{b} - \boldsymbol{a}) = 0$	A1
	$\Rightarrow \overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$, as $b \neq a$.	AG

[6]

32.	$\boldsymbol{a} \bullet \boldsymbol{b} = \lvert \boldsymbol{a} \rvert \, \lvert \boldsymbol{b} \rvert \cos \theta$	(M1)
	$\boldsymbol{a} \bullet \boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -2 \\ m \end{pmatrix} = 7 + 3m$	A1
	$\lvert \boldsymbol{a} \rvert = \sqrt{14} \quad \lvert \boldsymbol{b} \rvert = \sqrt{13 + m^2}$	A1
	$\lvert \boldsymbol{a} \rvert \lvert \boldsymbol{b} \rvert \cos \theta = \sqrt{14}\sqrt{13 + m^2} \cos 30°$	
	$7 + 3m = \sqrt{14}\sqrt{13 + m^2} \cos 30°$	A1
	$m = 2.27$, $m = 25.7$	A1A1

[6]

33. (a) $\begin{pmatrix} 1 & 2 & -3 & k \\ 3 & 1 & 2 & 4 \\ 5 & 0 & 7 & 5 \end{pmatrix}$ M1

$R_1 - 2R_2$

$\begin{pmatrix} -5 & 0 & -7 & k-8 \\ 3 & 1 & 2 & 4 \\ 5 & 0 & 7 & 5 \end{pmatrix}$ (A1)

$R_1 + R_3$

$\begin{pmatrix} 0 & 0 & 0 & k-3 \\ 3 & 1 & 2 & 4 \\ 5 & 0 & 7 & 5 \end{pmatrix}$ (A1)

Hence no solutions if $k \in \mathbb{R}, k \neq 3$ A1

(b) Two planes meet in a line and the third plane is parallel to that line. A1

[5]

34. (a) $x = 3 + 2m$
$y = 2 - m$
$z = 7 + 2m$ A1

$x = 1 + 4n$
$y = 4 - n$
$z = 2 + n$ A1

(b) $3 + 2m = 1 + 4n \Rightarrow 2m - 4n = -2$(i)
$2 - m = 4 - n \Rightarrow m - n = -2$(ii) M1
$7 + 2m = 2 + n \Rightarrow 2m - n = -5$(iii)
(iii) – (ii) $\Rightarrow m = -3$ A1
$\Rightarrow n = -1$ A1
Substitute in (i), $-6 + 4 = -2$. Hence lines intersect. R1
Point of intersection A is $(-3, 5, 1)$ A1

(c) $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & 2 \\ 4 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}$ M1A1

$\boldsymbol{r} \bullet \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}$ (M1)

$\boldsymbol{r} \bullet \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} = 29$

$x + 6y + 2z = 29$ A1

Note: Award M1A0 if answer is not in Cartesian form.

(d) $x = -8 + 3\lambda$
$y = -3 + 8\lambda$ (M1)
$z = 2\lambda$
Substitute in equation of plane.
$-8 + 3\lambda - 18 + 48\lambda + 4\lambda = 29$ M1
$55\lambda = 55$
$\lambda = 1$ A1
Coordinates of B are (–5, 5, 2) A1

(e) Coordinates of C are $\left(-4, 5, \frac{3}{2}\right)$ (A1)

$\boldsymbol{r} = \begin{pmatrix} -4 \\ 5 \\ \frac{3}{2} \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}$ M1A1

Note: Award M1A0 unless candidate writes $\boldsymbol{r}$ = or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

[18]

35. EITHER

Let s be the distance from the origin to a point on the line, then

$s^2 = (1 - \lambda)^2 + (2 - 3\lambda)^2 + 4$ (M1)

$= 10\lambda^2 - 14\lambda + 9$ A1

$\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda} = 20\lambda - 14$ A1

For minimum $\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda} = 0, \Rightarrow \lambda = \frac{7}{10}$ A1

OR

The position vector for the point nearest to the origin is perpendicular to the direction of the line. At that point:

$\begin{pmatrix} 1-\lambda \\ 2-3\lambda \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} = 0$ (M1)A1

Therefore, $10\lambda - 7 = 0$ A1

Therefore, $\lambda = \frac{7}{10}$ A1

THEN

$x = \frac{3}{10}, y = -\frac{1}{10}$ (A1)(A1)

The point is $\left(\frac{3}{10}, \frac{-1}{10}, 2\right)$. N3

[6]

36. (a)

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 3 & 5 & -1 & 0 \\ 1 & -5 & 2-a & 9-a^2 \end{array}\right]$$ M1

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & -8 & 3-a & 9-a^2 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$$ (M1)

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -a-1 & 9-a^2 \end{array}\right] \begin{array}{l} R_2 \to R_2 \times -\frac{1}{2} \\ R_3 \to R_3 - 2R_2 \end{array}$$ M1

When $a = -1$ the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 8 \end{array}\right]$$ A1

Hence the system is inconsistent $\Rightarrow a \neq -1$ R1

(b) When $a \neq -1$, $(-a-1)z = 9 - a^2$

$(a+1)z = a^2 - 9$

$\therefore z = \dfrac{a^2-9}{a+1}$ M1A1

$2y - z = 0 \Rightarrow y = \dfrac{1}{2}z = \dfrac{a^2-9}{2(a+1)}$ M1A1

$x = -3y + z = \dfrac{-3(a^2-9)}{2(a+1)} + \dfrac{2(a^2-9)}{2(a+1)} = \dfrac{9-a^2}{2(a+1)}$ M1A1

The unique solution is $\left(\dfrac{9-a^2}{2(a+1)}, \dfrac{a^2-9}{2(a+1)}, \dfrac{a^2-9}{a+1}\right)$ when $a \neq -1$

(c) $2 - a = 1 \Rightarrow a = 1$ M1

$\therefore$ The solution is $\left(\dfrac{8}{4}, -\dfrac{8}{4}, -\dfrac{8}{2}\right)$ or $(2, -2, -4)$ A1

[13]

37. (a) $\overrightarrow{AB} = -\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{BC} = \boldsymbol{i} + \boldsymbol{j}$ A1A1

(b) $\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -3 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ M1

$= -\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ A1

(c) Area of ΔABC = $\frac{1}{2}\left|-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}\right|$ M1A1

$= \frac{1}{2}\sqrt{1+1+4}$

$= \frac{\sqrt{6}}{2}$ A1

(d) A normal to the plane is given by $\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{BC} = -\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ (M1)
Therefore, the equation of the plane is of the form $-x + y + 2z = g$
and since the plane contains A, then $-1 + 2 + 2 = g \Rightarrow g = 3$. M1
Hence, an equation of the plane is $-x + y + 2z = 3$. A1

(e) Vector $\boldsymbol{n}$ above is parallel to the required line.

Therefore, $x = 2 - t$ A1
$y = -1 + t$ A1
$z = -6 + 2t$ A1

(f) $x = 2 - t$
$y = -1 + t$
$z = -6 + 2t$
$-x + y + 2z = 3$
$-2 + t - 1 + t - 12 + 4t = 3$ M1A1
$-15 + 6t = 3$
$6t = 18$
$t = 3$ A1
Point of intersection $(-1, 2, 0)$ A1

(g) Distance = $\sqrt{3^2 + 3^2 + 6^2} = \sqrt{54}$ (M1)A1

(h) Unit vector in the direction of $\boldsymbol{n}$ is e = $\frac{1}{|\boldsymbol{n}|} \times \boldsymbol{n}$ (M1)

$= \frac{1}{\sqrt{6}}(-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$ A1

Note: –e is also acceptable.

(i) Point of intersection of *L* and *P* is (–1, 2, 0).

$\overrightarrow{DE} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$ (M1)A1

$\Rightarrow \overrightarrow{EF} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$ M1

$\Rightarrow$ coordinates of F are (–4, 5, 6) A1

[25]

38. (a) $L_1 : x = 2 + \lambda; y = 2 + 3\lambda; z = 3 + \lambda$ (A1)
$L_2 : x = 2 + \mu; y = 3 + 4\mu; z = 4 + 2\mu$ (A1)
At the point of intersection (M1)
$2 + \lambda = 2 + \mu$ (1)
$2 + 3\lambda = 3 + 4\mu$ (2)
$3 + \lambda = 4 + 2\mu$ (3)
From (1), $\lambda = \mu$ A1
Substituting in (2), $2 + 3\lambda = 3 + 4\lambda$
$\Rightarrow \lambda = \mu = -1$ A1
We need to show that these values satisfy (3). (M1)
They do because LHS = RHS = 2; therefore the lines intersect. R1
So P is (1, –1, 2). A1 N3

(b) The normal to Π is normal to both lines. It is therefore given by the vector product of the two direction vectors.

Therefore, normal vector is given by $\begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 1 \\ 1 & 4 & 2 \end{pmatrix}$ M1A1

$= 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ A2
The Cartesian equation of Π is $2x - y + z = 2 + 1 + 2$ (M1)
i.e. $2x - y + z = 5$ A1 N2

(c) The midpoint M of [PQ] is $\left(2, \frac{3}{2}, \frac{5}{2}\right)$. M1A1

The direction of $\overrightarrow{MS}$ is the same as the normal to Π, i.e. $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ (R1)

The coordinates of a general point R on $\overrightarrow{MS}$ are therefore

$\left(2 + 2\lambda, \frac{3}{2} - \lambda, \frac{5}{2} + \lambda\right)$ (M1)

It follows that $\overrightarrow{PR} = (1 + 2\lambda)\boldsymbol{i} + \left(\frac{5}{2} - \lambda\right)\boldsymbol{j} + \left(\frac{1}{2} + \lambda\right)\boldsymbol{k}$ A1A1A1

At S, length of $\overrightarrow{PR}$ is 3, i.e. (M1)

$(1 + 2\lambda)^2 + \left(\frac{5}{2} - \lambda\right)^2 + \left(\frac{1}{2} + \lambda\right)^2 = 9$ A1

$1 + 4\lambda + 4\lambda^2 + \frac{25}{4} - 5\lambda + \lambda^2 + \frac{1}{4} + \lambda + \lambda^2 = 9$ (A1)

$6\lambda^2 = \frac{6}{4}$ A1

$\lambda = \pm\frac{1}{2}$ A1

Substituting these values, (M1)

the possible positions of S are (3, 1, 3) and (1, 2, 2) A1A1 N2

[29]

39. (a) Finding **correct** vectors $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$ A1A1

Substituting correctly in scalar product $\overrightarrow{AB} \cdot \overrightarrow{AC} = 4(-3) + 3(1) - 1(1)$ A1

$= -10$ AG N0

(b) $\left|\overrightarrow{AB}\right| = \sqrt{26}$ $\left|\overrightarrow{AC}\right| = \sqrt{11}$ (A1)(A1)

Attempting to use scalar product formula, $\cos \mathrm{B\hat{A}C} = \frac{-10}{\sqrt{26}\sqrt{11}}$ M1

$= -0.591$ (to 3 s.f.) A1

$\mathrm{B\hat{A}C} = 126°$ A1 N3

[8]