1a. Prove that the equation $3x^2 + 2kx + k - 1 = 0$ has two distinct real roots for all values of $k \in \mathbb{R}$. *[4 marks]*

Markscheme

$\Delta = b^2 - 4ac = 4k^2 - 4 \times 3 \times (k-1) = 4k^2 - 12k + 12$ ***M1A1***

Note: Award ***M1A1*** if expression seen within quadratic formula.

EITHER

$144 - 4 \times 4 \times 12 < 0$ ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

(and cannot be always negative as $a > 0$)

OR

sketch of $y = 4k^2 - 12k + 12$ or $y = k^2 - 3k + 3$ not crossing the x-axis ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

OR

write Δ as $4(k - 1.5)^2 + 3$ ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

[4 marks]

1b. Find the value of k for which the two roots of the equation are closest together. *[3 marks]*

Markscheme

closest together when Δ is least ***(M1)***

minimum value occurs when $k = 1.5$ ***(M1)A1***

[3 marks]

2. One root of the equation $x^2 + ax + b = 0$ is $2 + 3\text{i}$ where $a,\ b \in \mathbb{R}$. Find the value of a and the value of b. *[4 marks]*

Markscheme

METHOD 1

substituting

$-5 + 12\text{i} + a(2 + 3\text{i}) + b = 0$ ***(A1)***

equating real or imaginary parts ***(M1)***

$12 + 3a = 0 \Rightarrow a = -4$ ***A1***

$-5 + 2a + b = 0 \Rightarrow b = 13$ ***A1***

METHOD 2

other root is $2 - 3\text{i}$ ***(A1)***

considering either the sum or product of roots or multiplying factors ***(M1)***

$4 = -a$ (sum of roots) so $a = -4$ ***A1***

$13 = b$ (product of roots) ***A1***

[4 marks]

3. The roots of a quadratic equation $2x^2 + 4x - 1 = 0$ are α and β. *[6 marks]*

Without solving the equation,

(a) find the value of $\alpha^2 + \beta^2$;

(b) find a quadratic equation with roots α^2 and β^2.

Markscheme

(a) using the formulae for the sum and product of roots:

$\alpha + \beta = -2$ ***A1***

$\alpha\beta = -\frac{1}{2}$ ***A1***

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ ***M1***

$= (-2)^2 - 2\left(-\frac{1}{2}\right)$

$= 5$ ***A1***

Note: Award ***M0*** for attempt to solve quadratic equation.

[4 marks]

(b) $(x - \alpha^2)(x - \beta^2) = x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2$ ***M1***

$x^2 - 5x + \left(-\frac{1}{2}\right)^2 = 0$ ***A1***

$x^2 - 5x + \frac{1}{4} = 0$

Note: Final answer must be an equation. Accept alternative correct forms.

[2 marks]

Total [6 marks]

4. Show that the quadratic equation $x^2 - (5 - k)x - (k + 2) = 0$ has two distinct real roots for all real values of k. *[4 marks]*

Markscheme

$\Delta = (5 - k)^2 + 4(k + 2)$ ***M1A1***

$= k^2 - 6k + 33$ ***(A1)***

$= (k - 3)^2 + 24$ which is positive for all k ***R1***

Note: Accept analytical, graphical or other correct methods. In all cases only award ***R1*** if a reason is given in words or graphically. Award ***M1A1A0R1*** if mistakes are made in the simplification but the argument given is correct.

[4 marks]

One form of *Schwarz's Inequality* states that for any 4 real numbers p, q, r, and s,

$$pr + qs \leq \sqrt{p^2 + q^2} \cdot \sqrt{r^2 + s^2}.$$

Prove Schwarz's Inequality using the following steps.

a. Let $f(x) = (px + r)^2 + (qx + s)^2$. Explain why, for any real number x, $f(x) \geq 0$.

a. $f(x)$ is a sum of two squares of real numbers.

b. Expand $f(x)$, and express the discriminant of $f(x) = 0$ in terms of p, q, r, and s. (Leave your answer in factored form.)

b. $p^2x^2 + 2prx + r^2 + q^2x^2 + 2qsx + s^2 =$
$(p^2 + q^2)x^2 + (2pr + 2qs)x + (r^2 + s^2)$;
$4(pr + qs)^2 - 4(p^2 + q^2)(r^2 + s^2)$

c. What does the fact that $f(x) \geq 0$ tell you about where the graph of $y = f(x)$ is situated in relation to the x-axis? What does this tell you about the roots of the equation $f(x) = 0$? What does it therefore tell you about the discriminant of the equation $f(x) = 0$?

c. The graph lies above or on the x-axis. There is at most one real root. The discriminant is less than or equal to 0.

d. Use your answer to the last question of part (c) to prove Schwarz's Inequality.

d. $4(pr + qs)^2 - 4(p^2 + q^2)(r^2 + s^2) \leq 0$;
$(pr + qs)^2 \leq (p^2 + q^2)(r^2 + s^2)$; taking the square root of both sides, you get Schwarz's inequality.