$7^n + 2$ is divisible by 3, $n \in \mathbb{Z}^+$.

P_n is: $7^n + 2$ is divisible by 3 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $7^1 + 2 = 9$ which is divisible by 3 $\therefore$ P_1 is true

(2) If P_k is true, then $7^k + 2 = 3A$ where $A \in \mathbb{Z}$

$$\begin{aligned}
\therefore \quad 7^{k+1} + 2 &= 7 \times 7^k + 2 \\
&= 7(3A - 2) + 2 \quad \{\text{using } P_k\} \\
&= 21A - 14 + 2 \\
&= 21A - 12 \\
&= 3(7A - 4) \quad \text{where } 7A - 4 \text{ is an integer as } A \text{ is an integer}
\end{aligned}$$

$\therefore$ $7^{k+1} + 2$ is divisible by 3

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}, \; n \in \mathbb{Z}^+.$$

P_n is: $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \ldots. + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$
for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1 \times 2 \times 3 = 6$, RHS $= \dfrac{1 \times 2 \times 3 \times 4}{4} = 6$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \ldots. + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

$$\begin{aligned}
\therefore \quad & 1 \times 2 \times 3 + 2 \times 3 \times 4 + \ldots. + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \quad \{\text{using } P_k\} \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + \frac{4(k+1)(k+2)(k+3)}{4} \quad \{\text{equalising denominators}\} \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4}
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

$1 + r + r^2 + r^3 + r^4 + \dots + r^{n-1} = \dfrac{1 - r^n}{1 - r}, \; n \in \mathbb{Z}^+, \; r \neq 1.$

P_n is: $1 + r + r^2 + r^3 + \dots . + r^{n-1} = \dfrac{1 - r^n}{1 - r}$ for all $n \in \mathbb{Z}^+$, $r \neq 1$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1$ and RHS $= \dfrac{1-r}{1-r} = 1$ as $r \neq 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1 + r + r^2 + r^3 + \dots . + r^{k-1} = \dfrac{1 - r^k}{1 - r}$

Now $1 + r + r^2 + r^3 + \dots . + r^{k-1} + r^k = \dfrac{1 - r^k}{1 - r} + r^k$ {using P_k}

$$= \frac{1 - r^k}{1 - r} + r^k \left(\frac{1 - r}{1 - r} \right) \quad \text{\{equalising denominators\}}$$

$$= \frac{1 - r^k + r^k - r^{k+1}}{1 - r}$$

$$= \frac{1 - r^{k+1}}{1 - r}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

$5^{2n} - 1$ is divisible by 24, $n \in \mathbb{Z}^+$.

P_n is: $5^{2n} - 1$ is divisible by 24 for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $5^2 - 1 = 25 - 1 = 24$ is divisible by 24 $\therefore$ P_1 is true

(2) If P_k is true, then $5^{2k} - 1 = 24A$ where $A \in \mathbb{Z}$

Now $5^{2(k+1)} - 1 = 5^{2k}5^2 - 1$

$$= 25[24A + 1] - 1 \quad \text{\{using } P_k\text{\}}$$

$$= 25 \times 24A + 25 - 1$$

$$= 25 \times 24A + 24$$

$$= 24(25A + 1) \quad \text{where } 25A + 1 \text{ is an integer}$$

$\therefore$ $5^{2(k+1)} - 1$ is divisible by 24

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

$5^n \geq 1 + 4n, \quad n \in \mathbb{Z}^+.$

P_n is: $5^n \geqslant 1 + 4n$ for all $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, we have $5^1 \geqslant 1 + 4(1)$

$\therefore \quad 5 \geqslant 5$ which is true, so P_1 is true

(2) If P_k is true, then $5^k \geqslant 1 + 4k$

$$\begin{aligned} \text{Now} \quad 5^{k+1} = 5 \times 5^k &\geqslant 5 \times (1 + 4k) && \{\text{using } P_k\} \\ &\geqslant 5 + 20k \\ &\geqslant 5 + 4k && \{k \geqslant 0\} \\ &\geqslant 1 + 4(k+1) \end{aligned}$$

$\therefore \quad P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

If $u_1 = 1$ and $u_{n+1} = 3u_n + 2^n$,
then $u_n = 3^n - 2^n, \quad n \in \mathbb{Z}^+.$

P_n is: if $u_1 = 1$ and $u_{n+1} = 3u_n + 2^n$ for all $n \in \mathbb{Z}^+$, then $u_n = 3^n - 2^n$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $u_1 = 1 = 3^1 - 2^1$, so P_1 is true

(2) If P_k is true, then $u_k = 3^k - 2^k$ and

$$\begin{aligned} u_{k+1} &= 3u_k + 2^k \\ &= 3(3^k - 2^k) + 2^k && \{\text{using } P_k\} \\ &= 3^{k+1} - 3 \times 2^k + 2^k \\ &= 3^{k+1} - 2 \times 2^k \\ &= 3^{k+1} - 2^{k+1} \end{aligned}$$

$\therefore \quad P_{k+1}$ is true.

Since P_1 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}