Notes: Induction

Tuesday, August 7, 2018 8:05 AM

The process of formulating a general result from a close examination of the simplest cases is called **mathematical induction**.

For example: the first positive even number is $2 = 2 \times 1$
the second positive even number is $4 = 2 \times 2$
the third positive even number is $6 = 2 \times 3$
the fourth positive even number is $8 = 2 \times 4$

and from these results we *induce* that "the nth positive even number is $2 \times n$ or $2n$".

The statement that "the nth positive even number is $2n$" is a summary of the observations of the simple cases $n = 1, 2, 3, 4$ and is a statement which we *believe* is true. We call such a statement a **conjecture** or **proposition**.

Consider the sum of the first n odd numbers:

$$\begin{aligned} 1 &= 1 = 1^2 \\ 1+3 &= 4 = 2^2 \\ 1+3+5 &= 9 = 3^2 \\ 1+3+5+7 &= 16 = 4^2 \\ 1+3+5+7+9 &= 25 = 5^2 \end{aligned}$$

We may conjecture that "the sum of the first n odd numbers is n^2".

$P_n: \sum_{i=1}^{n} (2i-1) = n^2, \ \forall\ n \in \mathbb{Z}^+$

Step 1 (Base case)

$P_1: \sum_{i=1}^{1} (2i-1) = 2(1)-1 = 1 \Rightarrow 1^2 = 1$

$\therefore P_n$ is true for $n=1$

Step 2 (Assume P_n is true for some $n = k$)

a) $P_k: \sum_{i=1}^{k} (2i-1) = k^2$

b) Consider $n = \underline{k+1}$ (induction)

$$\begin{aligned} \sum_{i=1}^{k+1} (2i-1) &= \underbrace{1 + 3 + 5 + \cdots + (2k-1)} + \underline{(2(k+1)-1)} \\ &= \sum_{i=1}^{k} (2i-1) + 2(k+1) - 1 \\ &= k^2 + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

P_k true $\Rightarrow P_{k+1}$ true

$\therefore$ Since P_1 is true, and for arbitrary $k \in \mathbb{Z}^+$ $P_k \Rightarrow P_{k+1}$, then P_n true by P.M.I.

by P. M. I.

PROPOSITION NOTATION

We use P_n to represent a proposition which is defined for every integer n where $n \geqslant a$, $a \in \mathbb{Z}$.

THE PRINCIPLE OF MATHEMATICAL INDUCTION

The **principle of mathematical induction** constitutes a formal proof that a particular proposition is true.

> Suppose P_n is a proposition which is defined for every integer $n \geqslant a$, $a \in \mathbb{Z}$.
> If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geqslant a$.

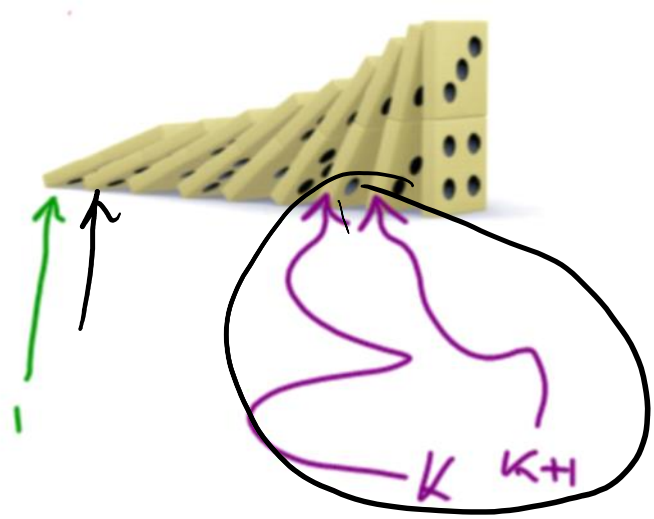
Suppose for a given proposition that P_1 is true. If we can show that P_{k+1} is true whenever P_k is true, then the truth of P_1 implies that P_2 is true, which implies that P_3 is true, which implies that P_4 is true, and so on.

One can liken the principle of mathematical induction to the **domino effect**. We imagine an infinite set of dominoes all lined up.

Provided that:

- the first domino topples to the right, and
- the $(k+1)$th domino will topple if the kth domino topples,

then given sufficient time, every domino will topple.



DIVISIBILITY

Consider the expression $4^n + 2$ for $n = 0, 1, 2, 3, 4, 5, \ldots$

$4^0 + 2 = 3 = 3 \times 1$
$4^1 + 2 = 6 = 3 \times 2$
$4^2 + 2 = 18 = 3 \times 6$
$4^3 + 2 = 66 = 3 \times 22$
$4^4 + 2 = 258 = 3 \times 86$

We observe that each of the answers is divisible by 3 and so we make the conjecture

$4^n + 2$ is divisible by 3 for all $n \in \mathbb{Z}$

divides evenly into

$P_n: \; 3 \mid 4^n + 2 \quad \forall n \in \mathbb{Z}, n \geq 0$

base case (P_0)

If $n = 0$, $4^0 + 2 = 3 = 1 \cdot 3$

$\therefore P_0$ is true

Assume P_n is true for some $n = k$

then $4^k + 2 = 3A$ where $A \in \mathbb{Z}^+$

$4^k = 3A - 2$ ✓

Consider $4^{k+1} + 2$ (P_{k+1})

$$\begin{aligned} 4^{k+1} + 2 &= 4(4^k) + 2 \\ &= 4(3A - 2) + 2 \\ &= 12A - 8 + 2 \\ &= 12A - 6 \\ &= 3(4A - 2), \end{aligned}$$

where $4A - 2$ is an integer since $A \in \mathbb{Z}^+$.

$\therefore 4^{k+1} + 2$ is divisible by 3 if $4^k + 2$ is divisible by 3.

Since P_0 is true, and P_k true $\Rightarrow P_{k+1}$ true then P_n is true $\forall n \in \mathbb{Z}, n \geq 0$.

Prove that $4^n + 2$ is divisible by 3 for $n \in \mathbb{Z}$, $n \geqslant 0$,

P_n is: $4^n + 2$ is divisible by 3 for $n \in \mathbb{Z},\ n \geqslant 0$.

Proof: (By the principle of mathematical induction)

(1) If $n = 0$, $4^0 + 2 = 3 = 1 \times 3$ $\therefore$ P_0 is true.

(2) If P_k is true, then $4^k + 2 = 3A$ where A is an integer, and $A \geqslant 1$.

Now
$$\begin{aligned} 4^{k+1} + 2 &= 4^1 4^k + 2 \\ &= 4(3A - 2) + 2 \quad \{\text{using } P_k\} \\ &= 12A - 8 + 2 \\ &= 12A - 6 \\ &= 3(4A - 2) \quad \text{where } 4A - 2 \text{ is an integer as } A \in \mathbb{Z}. \end{aligned}$$

Thus $4^{k+1} + 2$ is divisible by 3 if $4^k + 2$ is divisible by 3.

Since P_0 is true, and P_{k+1} is true whenever P_k is true,
then P_n is true for all $n \in \mathbb{Z},\ n \geqslant 0$ {Principle of mathematical induction}