Solutions!

INDUCTION PRACTICE AUGUST 15

Use the principle of mathematical induction to prove each.

1 A sequence is defined by $u_n = 2\times 3^{n-1}$.

Use the principle of mathematical induction to prove that
$u_1 + u_2 + \ldots + u_n = 3^n - 1$. *[6 marks]*

$P_n: \ S_n = \sum_{r=1}^{n} u_r = 3^n - 1$

Base case: For $n=1$: $S_1 = u_1 = 2 = 3^1 - 1$

$\therefore P_n$ true for $n=1$ (or say "$\therefore P_1$ true")

Inductive step: Assume P_k true for arbitrary k

So, $S_k = 3^k - 1$

Consider P_{k+1}:

$$S_{k+1} = (S_k) + [u_{k+1}]$$
$$= (3^k - 1) + [2\times 3^k]$$
$$= 3^k + 2\cdot 3^k - 1$$
$$= 3^k(1+2) - 1$$
$$= 3^k\cdot 3 - 1$$
$$= 3^{k+1} - 1$$

$\therefore P_k \rightarrow P_{k+1}$ true

$\therefore$ Since P_1 true, and since P_k true $\rightarrow P_{k+1}$ true, P_n is true by PMI.

2 Prove by induction that:

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

[6 marks]

$u_n = \frac{1}{n(n+1)}$ and $P_n: \ S_n = \sum_{r=1}^{n} u_r = \frac{n}{n+1}$

Base Case: (P_1)

For $n=1$: $S_1 = u_1 = \frac{1}{1(2)} = \frac{1}{2} = \frac{1}{1+1}$ $\therefore P_1$ true

Induction: Assume P_k true for arbitrary k

$S_k = \frac{k}{k+1}$

Consider P_{k+1}:

$$S_{k+1} = S_k + u_{k+1}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$
$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$
$$= \frac{(k+1)^2}{(k+1)(k+2)}$$
$$= \frac{k+1}{k+2}$$

$\therefore P_k$ true $\Rightarrow P_{k+1}$ true

Since P_1 true, and since $P_k \Rightarrow P_{k+1}$, P_n is true by P.M.I.

3 Prove by induction that:

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\ldots+\frac{1}{(2n-1)\times(2n+1)}=\frac{n}{2n+1}.$$ *[6 marks]*

$u_n = \frac{1}{(2n-1)(2n+1)}$ $\quad P_n: \; S_n = \sum_{r=1}^{n} u_r = \frac{n}{2n+1}$

Base case

$P_1: \quad S_1 = u_1 = \frac{1}{1\times3} = \frac{1}{3} = \frac{1}{2\times1+1} \quad \therefore P_n$ true for $n=1$

Inductive Step

- Assume statement is true for some arbitrary $n=k$: $S_k = \frac{k}{2k+1}$
- Consider P_{k+1}: $S_{k+1} = S_k + u_{k+1}$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$
$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$
$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$
$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$
$$= \frac{k+1}{2k+3}$$

$\therefore P_k$ true $\Rightarrow P_{k+1}$ true

$\boxed{\text{Since } P_1 \text{ true, and since } P_k \Rightarrow P_{k+1}, \; P_n \text{ is true by P.M.I.}}$

4 Prove that $1\times1!+2\times2!+3\times3!\ldots+n\times n!=(n+1)!-1$. *[6 marks]*

* Establish general term and proposition: $u_n = n\times n!$ and $P_n: \; S_n = \sum_{r=1}^{n} u_r = (n+1)!-1$

* Show base case: For $n=1$: $S_1 = u_1 = 1\times1! = 1 = 2!-1$

$\therefore$ the proposition is true for $n=1$.

* Inductive step:

First: Assume the statement is true for $n=k$;

... that is, $S_k = (k+1)!-1$

Second: Consider $n=k+1$

$$S_{k+1} = S_k + u_{k+1}$$
$$= (k+1)!-1 + (k+1)\times(k+1)!$$
$$= (k+1)!\,(1+(k+1))-1$$
$$= (k+2)(k+1)!-1$$
$$= (k+2)!-1$$

... so, if P_k is true for some arbitrary k, then P_{k+1} is true.

$\therefore$ Since P_1 is true and $P_k \Rightarrow P_{k+1}$, P_n is true by PMI.

5 Use the principle of mathematical induction to show:

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$ [6 marks]

$u_n = (-1)^{n-1} n^2$; P_n : $S_n = \sum_{r=1}^{n} u_r = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$

P_1 : $S_1 = u_1 = 1 = (-1)^0 \times \frac{1(2)}{2}$ $\therefore P_n$ true for $n=1$ (base case)

Assume P_n true for some arbitrary k.

P_k : $S_k = (-1)^{k-1} \cdot \frac{k(k+1)}{2}$

Consider P_{k+1}.

P_{k+1} : $S_{k+1} = S_k + u_{k+1}$

$$= \left[(-1)^{k-1}\right]\left[\frac{k(k+1)}{2}\right] + (-1)^k (k+1)^2$$

$$= (-1)^k (k+1)\left[-\frac{k}{2} + k + 1\right]$$

$$= (-1)^k (k+1)\left(\frac{k}{2} + 1\right)$$

$$= (-1)^k \frac{(k+1)(k+2)}{2}$$

$\therefore P_k$ true $\Rightarrow P_{k+1}$ also true

So, P_1 true, and $P_k \Rightarrow P_{k+1}$, $\therefore P_n$ true by P.M.I.

6 Prove that $(n+1)+(n+2)+(n+3)+\ldots+(2n) = \frac{1}{2}n(3n+1)$. [6 marks]

$u_n = n$; Proposition: $S_{2n} - S_n = \sum_{r=n+1}^{2n} u_r = \frac{1}{2} n(3n+1)$

* We are taking advantage of an opportunity to use a difference of two series to keep a formula simple.

Base case: For $n=1$: $S_2 - S_1 = u_2 = 2 = \frac{1}{2} \times 1 \times (3+1)$ $\therefore$ proposition is true for $n=1$.

Inductive step: Assume the statement is true for $n=k$, then

$$S_{2k} - S_k = \frac{1}{2}k(3k+1)$$

Consider $n=k+1$

$$S_{2(k+1)} - S_{k+1} = \frac{1}{2}(k+1)(3k+4)$$

$$S_{2k+2} - S_{k+1} = S_{2k} - S_k + u_{2k+1} + u_{2k+2} - u_{k+1}$$

$$= \frac{1}{2}k(3k+1) + (2k+1) + (2k+2) - (k+1)$$

(using the formulas for $S_{2k} - S_k$ and $u_{k+1}, u_{2k+1}, u_{2k+2}$)

$$= \frac{3}{2}k^2 + \frac{7}{2}k + 2$$

$$= \frac{1}{2}(3k^2 + 7k + 4)$$

$$= \frac{1}{2}(k+1)(3k+4)$$

$\therefore$ When the statement is true for $n=k$, it is also true for $n=k+1$

$\therefore$ Since P_1 true, and $P_k \to P_{k+1}$, P_n true by PMI.

7 Prove that $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$. *[6 marks]*

$u_n = n \times 2^n$ $\quad P_n: \quad S_n = \sum_{r=1}^{n} u_r = (n-1)2^{n+1} + 2$

$P_1: \quad S_1 = u_1 = 1 \times 2^1 = 2 = (1-1)2^2 + 2$ $\quad \therefore$ Proposition is true for $n=1$

Assume P_k true: $\quad S_k = (k-1)2^{k+1} + 2$

Consider P_{k+1}: $\quad S_{k+1} = S_k + u_{k+1}$

$$= (k-1)2^{k+1} + 2 + (k+1) \times 2^{k+1}$$
$$= 2^{k+1}(k-1+k+1) + 2$$
$$= 2k2^{k+1} + 2$$
$$= k2^{k+2} + 2$$

$\therefore P_k$ true $\Rightarrow P_{k+1}$ also true

The proposition is true for $n=1$, and if true for $n=k$, it is also true for $n=k+1$. Therefore P_n is true for all $n \in \mathbb{Z}^+$ by P.M.I.

8 Show that $5^n - 1$ is divisible by 4 for all $n \in \mathbb{N}$. *[8 marks]*

$P_n: \quad 4 \mid (5^n - 1) \quad \forall\, n \in \mathbb{N}$

P_0: (Base Case) $n=0$: $\quad 5^0 - 1 = 0 = 4 \times 0 \quad \therefore P_0$ true

P_k: Assume $\quad 5^k - 1 = 4A$ for some $A \in \mathbb{Z}$.

P_{k+1}: Consider $\quad 5^{k+1} - 1 = 5(5^k - 1) + 4$

$$= 5(4A) + 4$$
$$= 4(5A+1)$$

Since $5A+1 \in \mathbb{Z}$, $5^{k+1} - 1$ is divisible by 4.

$\therefore P_k$ true $\Rightarrow P_{k+1}$ true

Since proposition is true for $n=0$, and since P_k true $\Rightarrow P_{k+1}$ true, P_n is true by PMI.

9 Show that $4^n - 1$ is divisible by 3 for all $n \geq 1$. *[8 marks]*

P_n: $3 \mid (4^n - 1)$ $\forall n \geq 1$, $n \in \mathbb{Z}$

P_1: $4^1 - 1 = 3 = 3 \times 1$ $\therefore$ P_n true in base case, $n = 1$

Assume for some arbitrary $k \in \mathbb{Z}^+$, $k \geq 1$ that

$4^k - 1 = 3A$ for some $A \in \mathbb{Z}$

Consider $n = k+1$

$$
\begin{aligned}
4^{k+1} - 1 &= 4 \cdot 4^k - 1 \\
&= 4 \cdot 4^k - 4 + 3 \\
&= 4(4^k - 1) + 3 \\
&= 4(3A) + 3 \\
&= 3(4A + 1) \\
&= 3B \text{ where } B = 4A + 1 \in \mathbb{Z}
\end{aligned}
$$

$\therefore$ If the statement is true for $n = k$, then it is also true for $n = k+1$.

P_n is true for $n = 1$, and if true for $n = k$ it is also true for $n = k+1$. $\therefore$ P_n true by P.M.I.

10 Show that $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{N}$ integers. *[8 marks]*

Proposition: $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{N}$

note: More modern definitions of $\mathbb{N}$ include zero, such that $\mathbb{N} \equiv$ whole numbers

Base Case For $n = 0$: $7^0 - 3^0 = 0 = 4 \times 0$.

$\therefore$ the proposition is true for $n = 0$.

Inductive step

Assume the statement is true for $n = k$;

that is, $7^k - 3^k = 4A$ for some $A \in \mathbb{Z}$

Working towards: $7^{k+1} - 3^{k+1} = 4B$ for some $B \in \mathbb{Z}$

$$
\begin{aligned}
7^{k+1} - 3^{k+1} &= 7 \times (7^k - 3^k) + 4 \times 3^k \\
&= 7 \times 4A + 4 \times 3^k \quad \text{(using formula for } 7^k - 3^k\text{)} \\
&= 4 \times (7A + 3^k) \\
&= 4B \text{ where } B = 7A + 3^k \in \mathbb{Z}
\end{aligned}
$$

So, if the statement is true for $n = k$, then it is also true for $n = k+1$.

The proposition is true for $n = 0$, and if true for $n = k$ it is also true for $n = k+1$. Therefore, the proposition is true for all $n \in \mathbb{N}$ by the Principle of Mathematical Induction.

11 Use induction to prove that $30^n - 6^n$ is divisible by 12 for all integers $n \geq 0$. *[8 marks]*

12 Show using induction that $n^3 - n$ is divisible by 6 for all integers $n \geq 1$. *[9 marks]*

13 Using the principle of mathematical induction, prove that $n(n^2+5)$ is divisible by 6 for all integers $n \geq 1$. *[9 marks]*

14 Use induction to show that $7^n - 4^n - 3^n$ is divisible by 12 for all $n \in \mathbb{Z}^+$. *[9 marks]*

15 Prove, using the principle of mathematical induction, that $3^{2n+2} - 8n - 9$ is divisible by 64 for all positive integers n. *[8 marks]*

16 Show that the sum of the cubes of any three consecutive integers is divisible by 9. *[8 marks]*

17 Find the set of positive integers for which $3^n > n^3$ and prove your claim by induction. *[8 marks]*

Clearly for $n=3$, $3^n = n^3$

Proposition: $3^n > n^3 \quad \forall\, n \geq 4$

Base Case: For $n=4$: $3^4 = 81 > 4^3 = 64$ $\therefore$ Proposition is true for $n=4$.

Inductive step: Assume the statement is true for $n=k$ where $k \geq 4$; that is, $3^k > k^3$

Now... we are working towards $3^{k+1} > (k+1)^3$

$3^{k+1} = 3 \times 3^k > 3k^3$ (using $3^k > k^3$ from assumption)

$3k^3 = k^3 + k^3 + k^3$

For $k > 3$, $k^3 > 3k^2 > 3k+1$

$\therefore 3k^3 > k^3 + 3k^2 + 3k + 1 = (k+1)^3$ for $k > 3$

Hence $3^{k+1} > (k+1)^3$ for $k > 3$

So if the statement is true for $n=k$ then it is also true for $n=k+1$.

Proposition true for $n=4$, and being true for $n=k \geq 4$ means it's also true for $n=k+1$! $\therefore$ It's true for all $n \geq 4$ by P.M.I.

18 $2^n > 1 + n$ for all $n > 1$ *[8 marks]*

Base Case: For $n=2$: $2^2 = 4 > [1+2] = 3$ $\therefore P_n$ true for $n=2$

Inductive step: Assume P_k true, where $k \geq 2$; that is, $2^k > k+1$.

[we are now going to work towards showing $2^{(k+1)} > (k+1)+1$]

$2^{k+1} = 2 \times 2^k > 2(k+1)$ (using assumption P_k)

$\therefore 2^{k+1} > 2k+2$

and so also, $2^{k+1} > k+2$ for $k \geq 2$

{think like: $2^{k+1} > 2k+2 > k+2$ for $k \geq 2$}

So, if the statement is true for $n=k$, then it is also true for $n=k+1$.

$\therefore P_n$ is true for $n=2$, and if true for $n=k \geq 2$ it is also true for $n=k+1$. Therefore, P_n is true for all integers $n > 1$ by PMI.

19 $2^n > n^2$ for all $n \geq 4$ *[8 marks]*

20 $n! > 2^n$ for all $n \geq 4$ *[8 marks]*

Base Case: For $n = 4$: $4! = 24$ and $24 > 2^4$ because $2^4 = 16$.

Inductive step: Assume statement is true for $n = k$ where $k \geq 4$; that is, $k! > 2^k$

... Now, we try to show that if our assumption is true, then it implies the statement also is true for $k+1$. So, we are trying to reach: $(k+1)! > 2^{k+1}$

Consider $n = k+1$.

$$(k+1)! = (k+1) \times k! > (k+1) \times 2^k$$

Using assumption that $k! > 2^k$

$$k+1 > 2 \quad \text{for } k \geq 4$$

$$\Rightarrow (k+1) \times 2^k > 2^{k+1} \quad \text{for } k \geq 4$$

factor 2 out to validate

$$\therefore (k+1)! > 2^{k+1} \quad \text{for } k \geq 4$$

So, if the statement is true for $n = k$, then it's also true for $n = k+1$. $\therefore$ The statement is true $\forall n \in \mathbb{Z}, n \geq 4$ by PMI

21 $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}>\sqrt{n}$ for all $n>1$. *[8 marks]*

22 $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}>2(\sqrt{n+1}-1)$ for all $n\geq 1$. *[8 marks]*

23 Find the smallest integer N for which $3^N < N!$ Prove that $3^n < n!$ for all $n \geq$ N. [10 marks]

$3^0 = 1 = 0! = 0!$
$3^1 = 3 > 1! = 1$
$3^2 = 9 > 2! = 2$
$3^3 = 27 > 3! = 6$
$3^4 = 81 > 4! = 24$
$3^5 = 243 > 5! = 120$
$3^6 = 729 > 6! = 720$
$3^7 = 2187 < 7! = 5040$
$\therefore N = 7$

P_n: Proposition: $3^n < n! \;\; \forall n \geq 7$

P_7: Base Case: For $n = 7$: $3^7 = 2187 < 7! = 5040$
$\therefore P_n$ true for $n = 7$

P_k: Induction: Assume statement true for arbitrary $n = k$ where $k \geq 7$; that is, $3^k < k!$

Consider P_{k+1} (Working towards showing $3^{k+1} < (k+1)!$)

$3^{k+1} = 3 \times 3^k < 3k!$ (using assumption P_k)

$k \geq 7 \Rightarrow k + 1 > 3$
$\Rightarrow 3k! < (k+1)k! = (k+1)!$

$\therefore 3^{k+1} < (k+1)!$; $\therefore P_k$ true $\Rightarrow P_{k+1}$ true
...and so P_n is true by PMI.

24 Show that $(1+x)^n \geq 1 + nx$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. [8 marks]