BELLWORK 8/31/18

Let $f(x) = \dfrac{4}{x+2}, x \neq -2$ and $g(x) = x - 1$.

If $h = g \circ f$, find

(a) $h(x)$;

(b) $h^{-1}(x)$, where h^{-1} is the inverse of h.

(a) $h(x) = g\left(\frac{4}{x+2}\right)$ (M1)

$= \frac{4}{x+2} - 1 \quad \left(= \frac{2-x}{2+x}\right)$ A1

(b) **METHOD 1**

$x = \frac{4}{y+2} - 1$ (interchanging x and y) M1

Attempting to solve for y M1

$(y + 2)(x + 1) = 4 \quad \left(y + 2 = \frac{4}{x+1}\right)$ (A1)

$h^{-1}(x) = \frac{4}{x+1} - 2 \quad (x \neq -1)$ A1

METHOD 2

$x = \frac{2-y}{2+y}$ (interchanging x and y) M1

Attempting to solve for y M1

$xy + y = 2 - 2x \quad (y(x + 1) = 2(1 - x))$ (A1)

$h^{-1}(x) = \frac{2(1-x)}{x+1} \quad (x \neq -1)$ A1

Note: In either **METHOD 1** or **METHOD 2** rearranging first and interchanging afterwards is equally acceptable.

prove:

If $ax^2+bx+c=0$ has roots α and β,

then $\alpha+\beta=-\frac{b}{a}$ and

$$\alpha\beta=\frac{c}{a}$$

1a. Prove that the equation $3x^2 + 2kx + k - 1 = 0$ has two distinct real roots for all values of $k \in \mathbb{R}$. *[4 marks]*

1b. Find the value of k for which the two roots of the equation are closest together. *[3 marks]*

2. One root of the equation $x^2 + ax + b = 0$ is $2 + 3\mathrm{i}$ where $a,\ b \in \mathbb{R}$. Find the value of a and the value of b. *[4 marks]*

3. The roots of a quadratic equation $2x^2 + 4x - 1 = 0$ are α and β. *[6 marks]*
Without solving the equation,
(a) find the value of $\alpha^2 + \beta^2$;
(b) find a quadratic equation with roots α^2 and β^2.

4. Show that the quadratic equation $x^2 - (5 - k)x - (k + 2) = 0$ has two distinct real roots for all real values of k . *[4 marks]*

1a. Prove that the equation $3x^2 + 2kx + k - 1 = 0$ has two distinct real roots for all values of $k \in \mathbb{R}$. *[4 marks]*

Markscheme

$\Delta = b^2 - 4ac = 4k^2 - 4 \times 3 \times (k - 1) = 4k^2 - 12k + 12$ ***M1A1***

Note: Award ***M1A1*** if expression seen within quadratic formula.

EITHER

$144 - 4 \times 4 \times 12 < 0$ ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

(and cannot be always negative as $a > 0$)

OR

sketch of $y = 4k^2 - 12k + 12$ or $y = k^2 - 3k + 3$ not crossing the x-axis ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

OR

write Δ as $4(k - 1.5)^2 + 3$ ***M1***

Δ always positive, therefore the equation always has two distinct real roots ***R1***

[4 marks]

1b. Find the value of k for which the two roots of the equation are closest together. *[3 marks]*

Markscheme

closest together when Δ is least ***(M1)***

minimum value occurs when $k = 1.5$ ***(M1)A1***

[3 marks]

2. One root of the equation $x^2 + ax + b = 0$ is $2 + 3\mathrm{i}$ where $a,\ b \in \mathbb{R}$. Find the value of a and the value of b. *[4 marks]*

Markscheme

METHOD 1

substituting

$-5 + 12\mathrm{i} + a(2 + 3\mathrm{i}) + b = 0$ ***(A1)***

equating real or imaginary parts ***(M1)***

$12 + 3a = 0 \Rightarrow a = -4$ ***A1***

$-5 + 2a + b = 0 \Rightarrow b = 13$ ***A1***

METHOD 2

other root is $2 - 3\mathrm{i}$ ***(A1)***

considering either the sum or product of roots or multiplying factors ***(M1)***

$4 = -a$ (sum of roots) so $a = -4$ ***A1***

$13 = b$ (product of roots) ***A1***

[4 marks]

3. The roots of a quadratic equation $2x^2 + 4x - 1 = 0$ are α and β. *[6 marks]*

Without solving the equation,

(a) find the value of $\alpha^2 + \beta^2$;

(b) find a quadratic equation with roots α^2 and β^2.

Markscheme

(a) using the formulae for the sum and product of roots:

$\alpha + \beta = -2$ ***A1***

$\alpha\beta = -\frac{1}{2}$ ***A1***

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ ***M1***

$= (-2)^2 - 2\left(-\frac{1}{2}\right)$

$= 5$ ***A1***

Note: Award ***M0*** for attempt to solve quadratic equation.

[4 marks]

(b) $(x - \alpha^2)(x - \beta^2) = x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2$ ***M1***

$x^2 - 5x + \left(-\frac{1}{2}\right)^2 = 0$ ***A1***

$x^2 - 5x + \frac{1}{4} = 0$

Note: Final answer must be an equation. Accept alternative correct forms.

[2 marks]

Total [6 marks]

4. Show that the quadratic equation $x^2 - (5 - k)x - (k + 2) = 0$ has two distinct real roots for all real values of k. *[4 marks]*

Markscheme

$\Delta = (5 - k)^2 + 4(k + 2)$ ***M1A1***

$= k^2 - 6k + 33$ ***(A1)***

$= (k - 3)^2 + 24$ which is positive for all k ***R1***

Note: Accept analytical, graphical or other correct methods. In all cases only award ***R1*** if a reason is given in words or graphically. Award ***M1A1A0R1*** if mistakes are made in the simplification but the argument given is correct.

[4 marks]

One form of *Schwarz's Inequality* states that for any 4 real numbers p, q, r, and s,

$$pr + qs \leq \sqrt{p^2 + q^2} \cdot \sqrt{r^2 + s^2}.$$

Prove Schwarz's Inequality using the following steps.

a. Let $f(x) = (px + r)^2 + (qx + s)^2$. Explain why, for any real number x, $f(x) \geq 0$.

b. Expand $f(x)$, and express the discriminant of $f(x) = 0$ in terms of p, q, r, and s. (Leave your answer in factored form.)

c. What does the fact that $f(x) \geq 0$ tell you about where the graph of $y = f(x)$ is situated in relation to the x-axis? What does this tell you about the roots of the equation $f(x) = 0$? What does it therefore tell you about the discriminant of the equation $f(x) = 0$?

d. Use your answer to the last question of part (c) to prove Schwarz's Inequality.

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