

BELLWORK 8/31/18

Let $f(x) = \frac{4}{x+2}$, $x \neq -2$ and $g(x) = x - 1$.

If $h = g \circ f$, find

(a) $h(x)$;

(b) $h^{-1}(x)$, where h^{-1} is the inverse of h .

$$(a) \quad h(x) = g\left(\frac{4}{x+2}\right) \quad (M1)$$

$$= \frac{4}{x+2} - 1 \quad \left(= \frac{2-x}{2+x} \right) \quad A1$$

(b) **METHOD 1**

$$x = \frac{4}{y+2} - 1 \quad (\text{interchanging } x \text{ and } y) \quad M1$$

Attempting to solve for y M1

$$(y+2)(x+1) = 4 \quad \left(y+2 = \frac{4}{x+1} \right) \quad (A1)$$

$$h^{-1}(x) = \frac{4}{x+1} - 2 \quad (x \neq -1) \quad A1$$

METHOD 2

$$x = \frac{2-y}{2+y} \quad (\text{interchanging } x \text{ and } y) \quad M1$$

Attempting to solve for y M1

$$xy + y = 2 - 2x \quad (y(x+1) = 2(1-x)) \quad (A1)$$

$$h^{-1}(x) = \frac{2(1-x)}{x+1} \quad (x \neq -1) \quad A1$$

Note: In either **METHOD 1** or **METHOD 2** rearranging first and interchanging afterwards is equally acceptable.

prove:

If $ax^2+bx+c=0$ has roots α and β ,

then $\alpha + \beta = -\frac{b}{a}$ and

$$\alpha\beta = \frac{c}{a}$$

1a. Prove that the equation $3x^2 + 2kx + k - 1 = 0$ has two distinct real roots for all values of $k \in \mathbb{R}$.

[4 marks]

1b. Find the value of k for which the two roots of the equation are closest together.

[3 marks]

2. One root of the equation $x^2 + ax + b = 0$ is $2 + 3i$ where $a, b \in \mathbb{R}$. Find the value of a and the value of b .

[4 marks]

3. The roots of a quadratic equation $2x^2 + 4x - 1 = 0$ are α and β .

[6 marks]

Without solving the equation,

(a) find the value of $\alpha^2 + \beta^2$;

(b) find a quadratic equation with roots α^2 and β^2 .

4. Show that the quadratic equation $x^2 - (5 - k)x - (k + 2) = 0$ has two distinct real roots for all real values of k .

[4 marks]

1a. Prove that the equation $3x^2 + 2kx + k - 1 = 0$ has two distinct real roots for all values of $k \in \mathbb{R}$.

[4 marks]

Markscheme

$$\Delta = b^2 - 4ac = 4k^2 - 4 \times 3 \times (k - 1) = 4k^2 - 12k + 12 \quad \text{MIAI}$$

Note: Award *MIAI* if expression seen within quadratic formula.

EITHER

$$144 - 4 \times 4 \times 12 < 0 \quad \text{MI}$$

Δ always positive, therefore the equation always has two distinct real roots *RI*

(and cannot be always negative as $a > 0$)

OR

sketch of $y = 4k^2 - 12k + 12$ or $y = k^2 - 3k + 3$ not crossing the x -axis *MI*

Δ always positive, therefore the equation always has two distinct real roots *RI*

OR

$$\text{write } \Delta \text{ as } 4(k - 1.5)^2 + 3 \quad \text{MI}$$

Δ always positive, therefore the equation always has two distinct real roots *RI*

[4 marks]

1b. Find the value of k for which the two roots of the equation are closest together.

[3 marks]

Markscheme

closest together when Δ is least *(MI)*

minimum value occurs when $k = 1.5$ *(MI)AI*

[3 marks]

2. One root of the equation $x^2 + ax + b = 0$ is $2 + 3i$ where $a, b \in \mathbb{R}$. Find the value of a and the value of b .

[4 marks]

Markscheme

METHOD 1

substituting

$$-5 + 12i + a(2 + 3i) + b = 0 \quad (A1)$$

equating real or imaginary parts $(M1)$

$$12 + 3a = 0 \Rightarrow a = -4 \quad A1$$

$$-5 + 2a + b = 0 \Rightarrow b = 13 \quad A1$$

METHOD 2

other root is $2 - 3i$ $(A1)$

considering either the sum or product of roots or multiplying factors $(M1)$

$$4 = -a \text{ (sum of roots) so } a = -4 \quad A1$$

$$13 = b \text{ (product of roots) } \quad A1$$

[4 marks]

3. The roots of a quadratic equation $2x^2 + 4x - 1 = 0$ are α and β .

[6 marks]

Without solving the equation,

- (a) find the value of $\alpha^2 + \beta^2$;
(b) find a quadratic equation with roots α^2 and β^2 .

Markscheme

(a) using the formulae for the sum and product of roots:

$$\alpha + \beta = -2 \quad \mathbf{A1}$$

$$\alpha\beta = -\frac{1}{2} \quad \mathbf{A1}$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \quad \mathbf{M1}$$

$$= (-2)^2 - 2\left(-\frac{1}{2}\right)$$

$$= 5 \quad \mathbf{A1}$$

Note: Award **M0** for attempt to solve quadratic equation.

[4 marks]

$$(b) \quad (x - \alpha^2)(x - \beta^2) = x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 \quad \mathbf{M1}$$

$$x^2 - 5x + \left(-\frac{1}{2}\right)^2 = 0 \quad \mathbf{A1}$$

$$x^2 - 5x + \frac{1}{4} = 0$$

Note: Final answer must be an equation. Accept alternative correct forms.

[2 marks]

Total [6 marks]

4. Show that the quadratic equation $x^2 - (5 - k)x - (k + 2) = 0$ has two distinct real roots for all real values of k . [4 marks]

Markscheme

$$\Delta = (5 - k)^2 + 4(k + 2) \quad \text{MIAI}$$

$$= k^2 - 6k + 33 \quad \text{(AI)}$$

$$= (k - 3)^2 + 24 \text{ which is positive for all } k \quad \text{RI}$$

Note: Accept analytical, graphical or other correct methods. In all cases only award **RI** if a reason is given in words or graphically.

Award **MIAIAORI** if mistakes are made in the simplification but the argument given is correct.

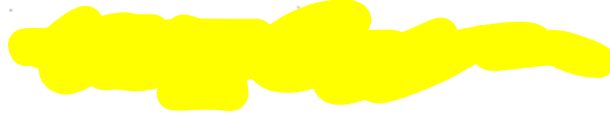
[4 marks]

One form of *Schwarz's Inequality* states that for any 4 real numbers $p, q, r,$ and $s,$

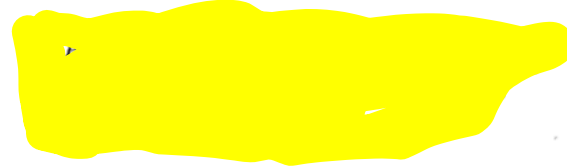
$$pr + qs \leq \sqrt{p^2 + q^2} \cdot \sqrt{r^2 + s^2}.$$

Prove Schwarz's Inequality using the following steps.

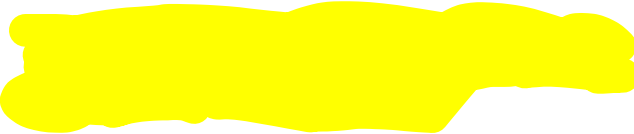
- a. Let $f(x) = (px + r)^2 + (qx + s)^2$. Explain why, for any real number $x,$ $f(x) \geq 0$.



- b. Expand $f(x),$ and express the discriminant of $f(x) = 0$ in terms of $p, q,$ $r,$ and $s.$ (Leave your answer in factored form.)



- c. What does the fact that $f(x) \geq 0$ tell you about where the graph of $y = f(x)$ is situated in relation to the x -axis? What does this tell you about the roots of the equation $f(x) = 0$? What does it therefore tell you about the discriminant of the equation $f(x) = 0$?



- d. Use your answer to the last question of part (c) to prove Schwarz's Inequality.

